CONVENTION, BACK-TO-SCHOOL ISSUE

The Mathematical Log

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'Elementary' Math Not Necessarily 'Easy' Victorian-Era Exam Problems Reveal

By Don Allen

(For Presentation at our Oklahoma Convention)

Creating, uncovering, solving, and sharing mathematical "problems" having just the right level of challenge and difficulty has to be among the most satisfying and personally rewarding of math-related individual and small-group activities. The "story problem" (something of a lost art form!), has special appeal in this connection, capturing as it can the mathematical essence of a more-or-less real-world situation, then offering it for interpretation and "model solution." "Lost art" or not, the story problem, at its best, we've always considered a very special—and difficult—literary form.

Nineteenth century Arithmetics and Algebras routinely incorporated a treasure trove of early story problems, indeed frequently featured them in extended groupings of miscellaneous or "promiscuous" exercises in which the harder story problems all but stole the show. Many of the very best of these story problems derived from competitive examinations, persisting in school and college textbooks from author (or "compiler") to author and from decade to decade. Such examination questions recall Elementary Algebra, even Arithmetic, as college subjects. Fractions, rates, and proportions! Linear relations, simultaneous equations, factorable quadratics! Such, in the main, was the mathematics to be brought to bear on such test questions. Elementary they were, but seldom easy!

Twelve of our favorite Victorian-era story problems follow, carefully chosen for their diversity, challenge, and lasting human interest. These particular questions have been selected for presentation at Oklahoma Convention and subsequent sharing with chapters ... from thousands of such story problems we've collected and filed over the years, from textbooks and other sources, North American and overseas.

Answers, as provided in the early schoolbooks, have been given at the conclusion of this presentation. Use them as you will! Required, of course, are full mathematical "solutions," building upon accepted and appropriate arithmetic, algebraic, and geometric insights and methodology and arriving inevitably at these numerical results. No "trial and error." No assuming "solutions in integers." Peeking at answers no doubt was resorted to by Victorian-era students—and schoolmasters—so perhaps shouldn't be wholly ruled out. Realize, however, that virtually all these story problems had their origin on competitive college examinations, time being limited and no "right answers" being around.

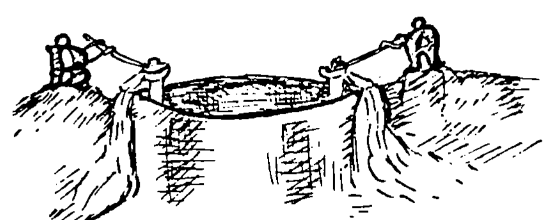
There follow our twelve favorite such story problems. Comment on solution is invited and encouraged. Interesting and unusual insights certainly will be shared. So set aside the hand-held calculator, ball-point pen, and other latter-day contrivances (not really!), transport yourself to a more leisurely, perhaps more reflective, bygone era, and try—really try!—for a first-time-around "12 out of 12."

I. On the Road to London

A and B travelled on the same road, and at the same rate, from Huntingdon to London. At the 50th mile stone from London A overtook a drove of geese, which were proceeding at the rate of three miles in two hours; and two hours afterwards he met a stage waggon which was proceeding at the rate of nine miles in four hours. B overtook the same drove of geese at the 45th mile stone, and met the same waggon exactly 40 minutes before he came to the 31st mile stone. Where was B when A reached London?

* * *

II. The Hold is Cleared



Art: Ali R. Amir-Moéz

The hold of a vessel partly full of water (which is uniformly increased by a leak), is furnished with two pumps worked by A and B, of whom A takes three strokes to two of B's; but four of B's throw out as much water as five of A's. Now, B works for the time in which A alone would have emptied the hold; A then pumps out the remainder, and the hold is cleared in 13 hours 20 minutes. Had they worked together, the hold would have been emptied in 3 hours 45 minutes, and A would have pumped out 100 gallons more than he did. Required the quantity of water in the hold at first, and the horary influx at the leak.

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The Mathematical Log

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Editorially Logged

The Convention--Back to School Issues of The Mathematical Log and Mathematical Tall Timbers (our Chapter supplement) are a special challenge and special pleasure for us--since we can be on the scene (this year in Oklahoma) when so many of you receive and can respond to your issues. We value the dialogue of Convention time, and would argue that it should and can be year-long and involve even more sponsors and members. Let's hear from you soon ... your priorities and your hopes for Mu Alpha Theta and for your Log.

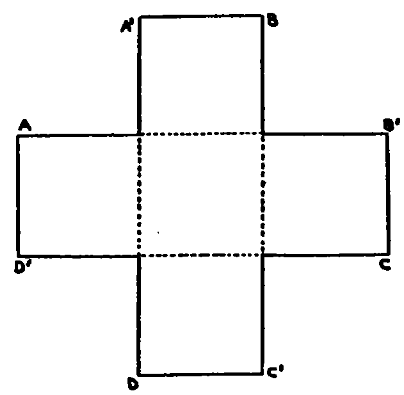
diaLOGue with the editor

The Texas Mu Alpha Theta Variable, Texas State's highly readable newsletter-plus, provided us with one of our more pleasurable mental workouts of the past year, and its challenge problem--new to us--seems one well worth sharing. To quote the challenge directly: "Using six ones, how many numbers between 1 and 100 can you write? You may use any order of operation, but you must use all six ones and no more than six." Giving ourselves a 24-hour time limit, we ruled out factorials (as not elementary, and not really needed), and used addition, subtraction, multiplication, division, brackets, surds, decimals, and repeating decimals, to represent (with exactly six ones) each of the first hundred numbers, then to extend our listing through 141. We delighted in such "elegant" (well, highly symmetrical) representations as

$$37 = \frac{111}{1+1+1} \quad \text{and} \quad 101 = \frac{1111}{11},$$

but were, and still are, stopped (somehow) by 142. Should you do better, and you well may, Texas State would, we're sure, be glad to hear from you, in care of Blessed Sacrament Academy, 1135 Mission Rd., San Antonio, TX 78210.

* * *



The third dimension, we find, "adds a dimension" to a mathematical problem in both a figurative and a literal sense. Featuring "a topless cubical box," an attractive little "space" question appeared recently in our favorite elementary "problem" department, that in School Science and Mathematics (Problem 3898, proposed March 1982, solved February 1983). The problem is by one of America's top "elementary problem" people, Charles W. Trigg, of San Diego. Reproduced with permission of SSM's editors, Trigg's "topless cubical box" challenge reads as follows:

"A topless cubical box with edges of length 1 has its vertical faces hinged to the horizontal square bottom. The top coinciding vertices of adjacent vertical faces (A and A', B and B', C and C', D and D' in the accompanying figure) are connected with pieces of string of length 1. The vertical faces are permitted to fall outward until stopped by the strings. What angles do the faces then make with the surface upon which the box rests?"

Problem solving, at the right challenge level, is both satisfying and good fun. SSM welcomes student solutions to the original problems which it includes in each issue.

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(Continued on page 6)

Victorian Problems

FROM PAGE ONE

III.
A Packet from Dover

A packet sailing from Dover with a fair wind arrives at Calais in two hours; and on its return, the wind being contrary, it proceeds six miles an hour slower than it went. Now, when it is half way over, the wind changing, it sails two miles an hour faster, and reaches Dover sooner than it would have done had the wind not changed, in the proportion of 6 : 7. Required the rates of sailing, and the distance between Dover and Calais.

* * *

IV.
To the End of the Voyage

A ship with a crew of 175 men set sail with a store of water sufficient to last to the end of the voyage. But in 30 days the scurvy made its appearance, and carried off three men every day; and at the same time a storm arose, which protracted the voyage three weeks. They were, however, just enabled to arrive in port without any diminution in each man's daily allowance of water. Required the time of the passage, and the number of men alive when the vessel reached harbour.

* * *

V.
From the Towns C and D

From the towns C and D, two travellers, A and B, set out to meet each other; A beginning his journey three hours sooner than B. They meet at the distance of 20 miles from D, and A reaches D one hour before B arrives at C. Next day, B having begun to return at an early hour, meets A who had then performed only 1/7 of his journey back; and, notwithstanding a subsequent delay of three hours, arrives at D soon enough, were it necessary, to go 28 miles farther before A reaches C. Required the distance between the towns, and the rate at which each person travels.

* * *

VI.
Two Sportsmen Went Shooting

Two sportsmen A and B went out shooting and brought home 10 birds. The sum of the squares of the number of shots was 2880, and the product of the numbers of shots fired by each was 48 times the product of the numbers of birds killed by each. If A had fired as often as B and B as often as A, then B would have killed 5 more birds than A: find the number of birds killed by each.

* * *

VII.
To Decipher a Manuscript

A certain student found it necessary to decipher an old manuscript. During previous experiences of the same kind he had observed that the number of words he could read daily varied jointly as the number of miles he walked and the number of hours he worked during the day. He therefore gradually increased the amount of daily exercise and daily work at the rate of 1 mile and 1 hour per day respectively, beginning the first day with his usual quantity. He found that the manuscript contained 232000 words, that he counted 12000 on the first day, and 72000 on the last day; and that by the end of half the time he had counted 62000 words: find his usual amount of daily exercise and work.

* * *

VIII.
Return of a Member

Fifty thousand voters, who have to return a member to an assembly, are divided into sections of equal size, and each section chooses an elector, the member being returned by the majority of such electors. There are two candidates, A and B. In those sections which return electors favourable to A, the majority is double the minority, while in those favourable to B, the minority forms only a tenth of the whole. After the primary elections a third candidate C comes forward, and is joined by so many electors of each party, that he is returned by a majority of 3 over A, and 14 over B. If C had not come forward, A would have been returned by a majority 19 less than the whole number of votes actually polled by C; and if the elections had been by the 50,000 voters directly between A and B, B would have had a majority of 6000. Find the number of sections.

* * *

IX.
The Sugar Plum Scuffle

A, in a scuffle, seized on 2/3 of a parcel of sugar plums, B catched 3/8 of it out of his hands, and C laid hold of 3/10 more; D ran off with all A had left, except 1/7 which E afterwards secured slyly for himself; then A and C jointly set upon B, who, in the conflict shed 1/2 he had, which were equally picked up by D and E, who lay perdue, B then kicked down C's hat, and to work they all went anew for what it contained; of which A got 1/4, B 1/2, D 2/7, C and E equal shares of what was left of that stock; D then struck 3/4 of what A and B last acquired, out of their hands; they with difficulty recovered 5/8 of it in equal shares again, but the other three carried off 1/5 apiece of the same. Upon this they called a truce, and agreed that the 1/3 of the whole left by A at first should be equally divided among them. How much of the prize, after this distribution, remained with each of the competitors?

* * *

X.
At a General Election

At a general election the whole number of Liberals returned was 15 more than the number of English Conservatives, the whole number of Conservatives was 5 more than twice the number of English Liberals. The number of Scotch Conservatives was the same as the number of Welsh Liberals, and the Scotch Liberal majority was equal to twice the number of Welsh Conservatives, and was to the Irish Liberal majority as 2 : 3. The English Conservative majority was 10 more than the whole number of Irish members. The whole number of members was 652, of whom 60 were returned by Scotch constituencies. Find the numbers of each party returned by England, Scotland, Ireland, and Wales, respectively.

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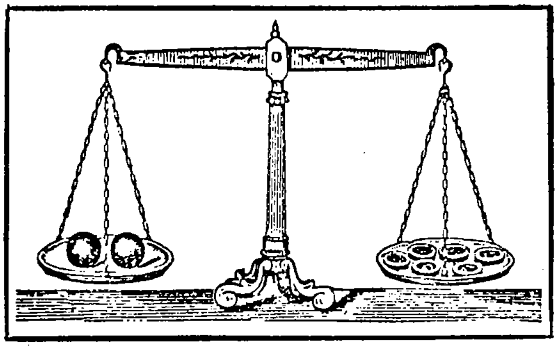
(Continued on page 4)

Victorian Problems

FROM PAGE THREE



THE CONCEPT OF EQUATION is well illustrated (by a two-pan "balance") on the first page of Tower's Intellectual Algebra (Boston, 1860), but pictures and diagrams are rare in early Arithmetics and Algebras. (Euclids occasionally gathered all necessary diagrams on one or more fold-out "plates.") Considerable challenge may be found in "story problems" from such early textbooks.

XI.
In Concentric Circles

Three men, A, B, C, walk in the same direction in the circumferences of three concentric circles, starting simultaneously from points where they are at their least distances from each other. A walks his circuit in an even number of hours (greater than four), B and C their circuits in one hour and two hours less respectively. Whenever A and B are at their greatest distance from each other, they alter their rates in such a manner, that the times they would take to walk their circuits at the rates they are then going are interchanged; and whenever A and C are again at their least distance their times are interchanged in a similar manner. When A and B are at their greatest distance the first time, A has walked a distance equal to 22 times C's circuit; and when they are at their greatest distance the third time, B has walked a distance equal to 42 times A's circuit, and C has then walked ten miles less than 40 times B's circuit, and is at his least distance from B. Required the rates of A, B, C, at first.

* * *

XII.
The Sewer being Insufficient

From the middle of a town two streets branched off, and crossed a river that ran a straight course, by two bridges, A and B. From their junction a sewer, equally inclined to both streets, led to a point in the river at the distance of six chains from the bridge A, and a distance from B less by eleven chains than the length of the sewer; the expense of making it amounting to as many pounds per chain as there were chains in the street leading to A. The sewer, however, being insufficient to carry off the water, an additional drain was made from a point in this street, distant four chains from the bridge A, which entered the river at the same point with the sewer, and was equally inclined to the river and the sewer. Now, it was found that a drain down the middle of each street, at the rate of 9 pounds per chain, would have cost only 54 pounds more than the expense of the sewer. Required the lengths of the streets and the sewer, and the distance of its mouth from the bridge B.

* * *

Bonus Question—for Sponsors
The Income of a Schoolmaster

The income of a schoolmaster arises partly from ten pupils residing in his house; and partly from an endowment, which produces a certain number of quarters of wheat each year. When wheat sells for 60 shillings the expenditure of his family (249 pounds) is less than his savings by a number, which when divided by twice the number of his pupils expresses the proportion which the clear gain bears to the whole charge for each pupil. In the following year wheat falls to 55 shillings, and a tax of 8 pence in the pound is laid upon income, payable upon the net income of the preceding year; but the cost of living for his pupils being diminished (so that, in fact, the amount of income-tax he has to pay, with 10 shillings added, would just support one pupil), he finds that his savings are greater than in the year previous by a sum equal to the difference of his net income in the two years, which is 1/18th of the expenditure of his family in the second year, besides allowing for an outlay of 15 pounds in repairs. The net income from pupils in the first year being 330 pounds, find that from the endowment in the same year and the ratio of the costs of living in the two years.

(A pre-decimal pound Sterling of 20 shillings, each of 12 pence, is of course, understood. Ed.)

* * *

Textbook Answers

Textbook answers provided for the twelve early "story problems" are as follows:

I. The question derives from a Cambridge college paper of 1815. It still found a place in school texts of 1894. Twenty-five miles from London.

II. Twelve hundred gallons in the hold; the influx was 120 gallons per hour.

III. Rates of sailing on the return passage were 5 and 7 miles per hour; distance, 22 miles.

IV. The voyage lasted 79 days; the number of men alive was 28.

V. The distance was 56 miles, and the rates 7 and 9 1/3 miles per hour.

VI. A shot 6 birds and B shot 4 birds.

VII. He walked 3 miles and worked 4 hours a day, or walked 4 miles and worked 3 hours a day.

VIII. One hundred.

IX. Figuring out the author's meaning is most of the problem, we suspect—and most of the fun.
A's part, 2863/26880 B's part, 6335/26880
C's part, 2438/26880 D's part, 10294/26880
E's part, 4950/26880

X. Sets up nicely...as eight simple (linear) relations in eight "unknowns." Conservatives: English 286, Scotch 19, Irish 35, Welsh 11. Liberals: English 173, Scotch 41, Irish 68, Welsh 19.

XI. Three, four, and five miles per hour, respectively.

XII. Length of streets, 18 and 30 chains; sewer, 21 chains; distance from B, 10 chains.

"Story problems" selected and discussed from "Sums of Yesteryear," a mathematics education feature column

Pappus Theorem of Antiquity Yields Instructive Insights

By Ali R. Amir-Moéz
Mathematics Editor

Pappus (Παππoσ), the famous Greek geometer of the third century (possibly first century) who lived in Alexandria, wrote his Mathematics collections, from which the last six books out of eight come down to us [1]. Among his works one finds a formula for the volume inside of certain surfaces of revolution. This was rediscovered or possibly reproved by Guldin. In this note we would like to experiment with Pappus' formula and verify it through other formulas.

Pappus' Theorem:

Let A be the area of a plane region P enclosed in a closed curve, and l a straight line in the plane of the region which is on one side of P (Fig. 1). Suppose

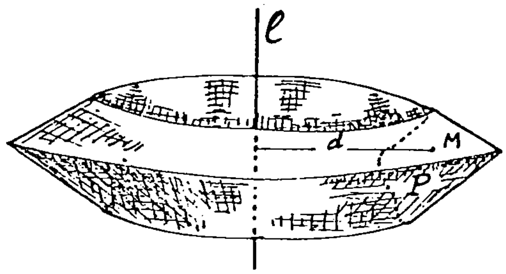


Figure 1

M is the centroid of P and d is the distance from M to l. Rotating the region about l, we obtain a solid. The volume V of this solid is A multiplied by the circumference of the circle that M traces about l. Thus we get

$$V = A(2\pi d). \qquad (1)$$

One may call (1) Pappus' formula.

Cylinders:

Let ABCD be a rectangle of dimensions a and b (Fig 2). It is clear that M, the centroid of it, is the

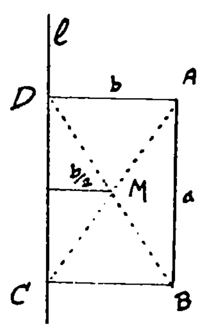


Figure 2

point of intersection of the diagonals. Now let us rotate the rectangle about l = DC. It is clear that the volume of the generated cylinder is

$$V = \pi b^2 a.$$

Next we apply Pappus' formula. It is apparent that the distance from M to l is $\frac{b}{2}$; that is, $d = \frac{b}{2}$. Applying (1) we obtain

$$V = (ab)\ (2\pi)\ (\tfrac{b}{2}) = \pi b^2 a.$$

Now let us move l to a distance p parallel to DC (Fig. 3). Rotating ABCD about l, we obtain a cylin-

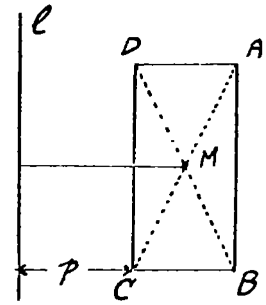


Figure 3

drical shell. Here again we apply (1), where A = ab, and $d = \frac{b}{2} + p$. So we get

$$V = (ab)\left[2\pi(\tfrac{b}{2} + p)\right] = \pi ab(b + 2p). \qquad (2)$$

Indeed, one may obtain this volume by subtracting the volume of the cylinder of radius p from the volume of the large cylinder of radius p + b, i.e.,

$$V = \pi(p + b)^2 a - \pi p^2 a. \qquad (3)$$

One can easily carry out the algebra and show that (3) is the same as (2).

Cones:

Let ABC be a right triangle, where B is the vertex of the right angle (Fig. 4). We rotate ABC about l = BC. Let M be the centroid.

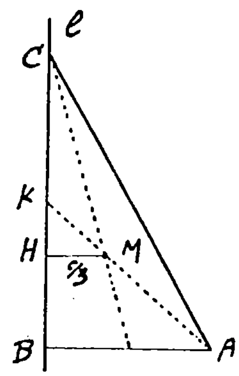


Figure 4

It is well known that M is the point of intersection of the three medians. Let AK be the median of the side BC. Then

$$AM = \frac{2}{3}\,AK, \quad \text{and } MK = \frac{1}{3}\,AK.$$

Indeed, a proof can be found in any book on elementary geometry. Now suppose MH = d is the distance from M to l, = BC. One observes that the triangles KMH and KAB are similar. Therefore

$$\frac{MH}{AB} = \frac{MK}{AK} = \frac{1}{3}.$$

(Continued on page 6)

Pappus Theorem

FROM PAGE FIVE

This implies that

$$d = MH = \frac{1}{3}(AB) = \frac{c}{3}.$$

Rotating the triangle ABC about BC, we obtain a cone. Now we shall apply (1). Note that $A = \frac{ac}{2}$, and $d = \frac{c}{3}$. So the volume will be

$$V = \left(\frac{ac}{2}\right)\left[2\pi\left(\frac{c}{3}\right)\right] = \frac{\pi c^2 a}{3}.$$

One may experiment with other flat objects; for example, rotating a triangle about one of its sides.

Tori:

A torus provides a good illustration of the application of Pappus' formula. Let l be a line in the plane of a circle of center M and radius a (Fig. 5). Let d be the distance from M to l such that $d \gtreqless a$.

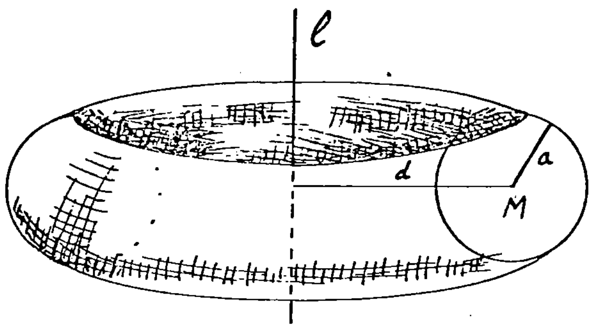


Figure 5

Then, applying (1), the volume of the torus obtained by rotating the circle about l will be

$$V = (\pi a^2)(2\pi d) = 2\pi^2 a^2 d.$$

One may, as a further example, rotate an ellipse about a line. This gives the opportunity to see that πab is the area of an ellipse whose half major and half minor axes are respectively a and b.

Obtaining Centroids:

Sometimes it will be useful to employ Pappus' formula to obtain the centroid. We shall give an example. Consider a half circle of radius a with diameter AB (Fig. 6). Let M be the centroid and d the distance

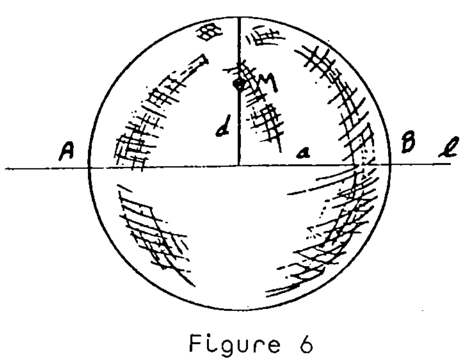


Figure 6

from M to AB. If we rotate the half circle about AB, we obtain a sphere of radius a. The volume can be written in two ways; by the use of the centroid, or by reference to the formula for the volume of a sphere of radius a; i.e., $\frac{4\pi a^3}{3}$ or by (1). Thus

$$V = \frac{4}{3}\pi a^3 = (2\pi d)\left(\frac{\pi a^2}{2}\right). \qquad (4)$$

From (4) we obtain

$$d = \frac{4a}{3\pi}.$$

This means that the centroid M of this semicircle is on the perpendicular bisector of AB, where the distance from M to AB is $\frac{4a}{3\pi}$.

Reference

[1] David Eugene Smith, History of Mathematics, 1925. Vol. I. Reprinted by Dover Publications, Inc.

diaLOGue . . . FROM PAGE TWO

The long-division "restoration challenge" (April Log, p. 4) provided an interesting exercise in logic (by our pencil and scrap paper approach), or else a good work-out for programmer and computer. Stephen Malinak, Grade X student at St. Vincent-St. Mary's High School, opted for the latter approach ... using his computer to multiply 1000 number pairs to come up with the unique result. "Computers are wonderful, aren't they!" exclaims Stephen. Yes, in their place ... and we also like Stephen's "smiling face" typewriter key.

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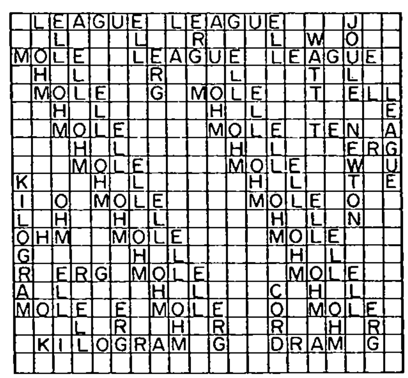
"Crossword Challenges"? Nothing seems to bring out the competitiveness, resourcefulness, and down-right ingenuity of Loggers to quite the extent of these mathword puzzlers, and "Mingled Measures," our February challenge, was no exception. Brian Schar of Pulaski High School chapter, Pulaski, WI, submitted 297 points of intricately interwoven metric and pre-metric terminology, as illustrated below. Will Brian's



297 record stand? Frankly, we don't know. The clever "staircasing" and overlapping are strong points in its favor,-- but (i) previous winners have been more highly symmetrical, (ii) the all-blank 20th row may leave room for improvement, (iii) the proportion is fairly high (34/73 = 46+%) of older, 3-point terms. So it's Logmaster Brian (congratulations!), with the "ongoing" challenge: can you (or, collectively, your chapter) better Brian's impressive 297 "Measures" score?

* * *

Sam Loyd, America's master puzzlist, would have enjoyed meeting Logmaster Daniel Paul Johnson, Bedford, PA. The Bedford chapter member took on Loyd's Columbus Puzzle (December Log), correctly surmised that the winning total would be an exact 82, and came up with eight solutions. Representatively,

$$82 = 80.\dot{4}\dot{7} + .\dot{5} + .\dot{9}\dot{6}$$