The Mathematical Log

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Challenge of a Lifetime for Puzzle Fans Highlights 1981 as 'Year of the Cube'

Sequences of Moves Seen Key to Solution

Unless something truly "out of this world," say the landing of a Space fleet or the proof of Fermat's Last Theorem, should happen between now and December 31st, then 1981 should go down in history--our kind of history--as the Year of the Cube. What Cube? The rotating, intuition-defying thingamabob from Hungary that this year has taken Planet Earth by storm and has had abstract mathematicians invoking group theory (and worse!) over what very possibly was bought as a child's toy.

Scientific American featured the Cube on its March 1981 cover, and Math-Jeunes, The Log's Belgian counterpart, termed it "peut-être le plus beau casse-tête de notre siècle" (perhaps the best puzzle of the century).

The Cube, as merchandised in North America, is likely to be Rubik's Cube, for Ernő Rubik, its Hungarian inventor. A similar device was patented by Terutoshi Ishige in Japan. Hofstadter, who did the Scientific American cover story, considers the Cube "spiritual kin" to American Sam Loyd's "15" sliding-block puzzle of 1873, perhaps the most successful puzzle of all time.

Superficially, the Cube suggests a 3x3x3 array of small "cubelets," the visible faces being of six colors. Indeed, when purchased, the Cube presents each face as nine identically colored squares. Faces rotate through quarter-, half-, and three-quarter-turns, thereby relocating and reorienting the colored cublets in literally quintillions of patterns.

Therein lies the Challenge: to get to a pre-selected pattern--or, more probably, to restore the Cube to the ultimate simplicity of the six one-color faces that it had when it was bought.

Hofstadter recognizes the Challenge in all its depth: "No one can restore a messed-up ... Cube to its pristine state by mere trial and error. Anyone who gets back to the Start position has built up a small science."

The Cube calls for careful initial examination. Its cubelets, it becomes apparent, are of three distinct types. Center cubelets (let's call them) display one face, are in the center of each nine-square face, and remain fixed in position relative to one another. Edge cubelets display two faces: there are 12 edge cubelets, each a different color-pair. Corner cubelets display three faces: there are 8 corner cubelets, each a different color-triple.

Dissection of a Cube--Scientific American shows how--reveals the diabolically clever internal mechanism, plus the fact that cubelets are not cubes but possess only the faces that they initially display. Hence there are 26 (6 + 12 + 8, not 3x3x3) cubelets, and center, edge, and corner cubelets maintain their roles through all transformations.

How does one "restore" a "messed-up" Cube? With great difficulty! Math-Jeunes does provide general guidance as to initial stages, then step-by-step advice for the difficult sequences of possible later moves. These good approaches The Log is pleased to share.

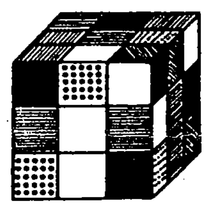

Fig. 1

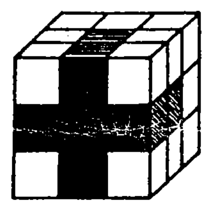

Fig. 2

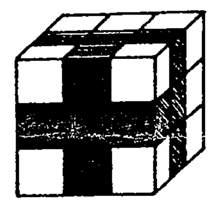

Fig. 3

Initial stages in the "restoring" of a Cube are depicted in Figs. 1-3, above. Fig. 1 represents an arbitrarily "messed-up" Cube. In Fig. 2, a "cross"

of a single color has been produced on one of the faces. Further, the four edge cubelets of this initial "cross" have, as their second color, the color of the center square next to which they lie. Fig. 3 extends the pattern, each face adjoining the face with the "cross" now having a middle line of the face's center color, parallel to the plane of the

(Continued on page 5)

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EDITORIAL

WELCOME BACK . . . TO YOUR LOG

One secret of a strong Mu Alpha Theta chapter and an especially successful club year is, we are convinced, a prompt and vigorous launch in September. Right from the start, sponsors and executives planning a new Mu Alpha Theta year are looking for ideas: to these, we like to think that The Mathematical Log can make a significant contribution. Hence this Fall Log will go to press in late Spring, and should be available at back-to-school time to help with initial Fall meetings.

Welcome, or welcome back, to The Mathematical Log and to Mu Alpha Theta! Let's be hearing from you. The Log, as we see it, is a vital link between chapters--we want to share news of programs and problems and personalities.

So, have a good Mu Alpha Theta year--one of the best--and resolve to share the good things through the columns of your Log. *H.D.A.*

TRY THIS

POLYHEDRAL LAMPSHADES CONSTRUCTION CHALLENGE

"Polyhedral lampshades can be obtained in fashionable arty-crafty shops (at fashionable arty-crafty prices), but can be constructed very easily and cheaply." So states Mathematical Digest, University of Cape Town, which goes on to illustrate general principles in a plan for a dodecahedral lampshade.

Care will be needed in selecting materials, of course, and advice of someone experienced in crafts might be sought. But the mathematics is straightforward. Cut out of thin card twelve circles of radius 10 cm and inscribe a regular pentagon in each. Use a protractor for the required 108° angle. Score lightly along the edges of the pentagon, and cut small rectangular pieces out of each corner (Fig. 1).

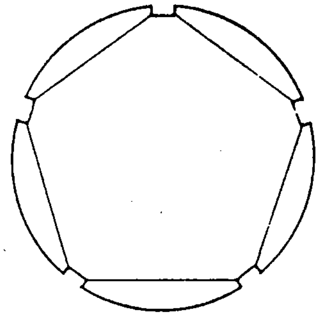


Fig. 1

In one of the twelve pieces cut six holes, as shown in Fig. 2. This piece will become the top face of the lampshade. The central hole is for the wire, and the other five holes provide ventilation. In another piece cut out a single circle of radius 6 cm. This will be the bottom face (Fig. 3).

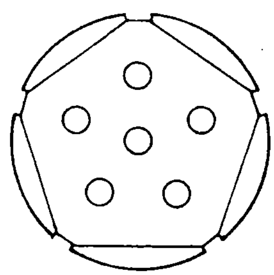


Fig. 2

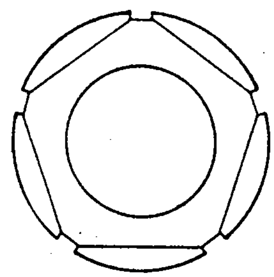


Fig. 3

Now join the pieces together to form a regular dodecahedron (Fig. 4). The flaps, which are not shown in the picture, should point outwards. The faces can be secured with elastic bands temporarily, but a clear adhesive is a better permanent arrangement.

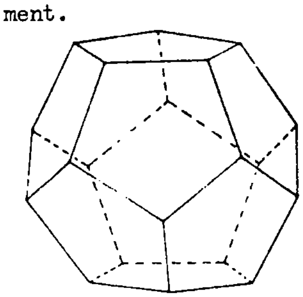


Fig. 4

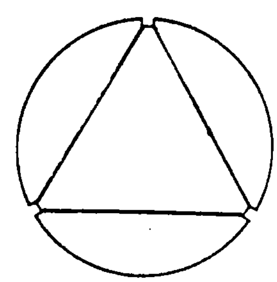


Fig. 5

Several other polyhedra make excellent lampshades. Triangular faces (as in Fig. 5) suggest octahedral and icosahedral forms.

'Puzzles' May Prepare for More Serious Problems Pólya Argues in Presenting, Discussing Solutions

(Solutions and notes prepared by Professor Pólya for two additional "puzzles" reprinted in our 25th Birthday Log (Spring 1981)follow. The solutions first appeared in The Log for April 1960. Professor Pólya has urged in connection with these problems: "Behind the diversity of details, try to perceive the outline of a common procedure." Ed.)

Al, Bill and Chris planned a big picnic. Each boy spent 9 dollars. Each bought sandwiches, ice cream and soda pop. For each of these items, the boys spent jointly 9 dollars, although each boy split his money differently and no boy spent the same amount of money for two different items. The greatest single expense was what Al paid for ice cream; Bill spent twice as much for sandwiches as for ice cream. How much did Chris pay for soda pop? (All amounts are in round dollars.)

Let x, y and z denote the amounts that one of the boys spent for the three different items.

MAJOR CONDITION: x, y and z are integers, $0 < x < y < z$ and

$$x + y + z = 9.$$

We observe first that x cannot be equal to 3 or greater, since $3 + 4 + 5 = 12 > 9$. And so we easily find that there are precisely three solutions:

$$2 + 3 + 4 = 9,$$
$$1 + 2 + 6 = 9,$$
$$1 + 3 + 5 = 9.$$

Let us, however, rearrange the left hand sides so that the first amount on each line is spent for the same kind of merchandise, similarly for the second, and (consequently) for the last number. Hence the

COMPLEMENT TO THE MAJOR CONDITION: Rearrange the above three equations so that the numbers in each of the three columns yield the sum 9.

This condition is satisfied by the following array:

$$3 + 4 + 2 = 9,$$
$$1 + 2 + 6 = 9,$$
$$5 + 3 + 1 = 9,$$

and it is easy to see that the given arrangement is the only arrangement satisfying the complement to our major condition. (Keep the second line, as you may, unchanged and see what you can add to the smallest number 1, or to the largest 6, of this line to get the desired sum 9.)

MINOR CONDITIONS: The greatest single expense is 6: therefore, the second line represents Al's expenses and the third column the expenses for ice cream. The only number in the array that is double the number on the same line in the third column is 4: therefore, line 1 is for Bill and column two for sandwiches. As the other possibilities are eliminated, we conclude that the number in the last line and the first column, 5, is what Chris paid for soda pop.

* * *

In preparation for Hallowe'en, three married couples, the Browns, the Joneses, and the Smiths, bought "treats" for the neighborhood youngsters. Each bought as many identical treats as he (or she) paid cents for one of them. Each wife spent 75 cents more than her husband. Ann bought one more present than Bill Brown, Betty one less present than Joe Jones. What is Mary's last name?

Let x and y denote the number of "treats" bought by a wife and her husband, respectively.

MAJOR CONDITION:

$$x^2 - y^2 = 75.$$

Now, the number $75 = 3 \times 5 \times 5$ has just 6 different divisors

$$1, 3, 5, 15, 25, 75$$

and so

$$(x - y)(x + y) = 1 \times 75 \text{ or } 3 \times 25 \text{ or } 5 \times 15$$

and we have three possibilities

$x - y = 1$	$x - y = 3$	$x - y = 5$
$x + y = 75$	$x + y = 25$	$x + y = 15.$

Solving the three systems of equations, we exhibit the three alternatives for the numbers of treats bought:

wife	husband
38	37
14	11
10	5

The MINOR CONDITIONS permit us to identify conclusively that

Ann	38	37	Bill Brown
	14	11	Joe Jones
Betty	10	5	

As the other alternatives are eliminated, the last name of Mary must be Jones.

\- - -

THE PATTERN

You may ask: "What is the use of solving such puzzles?" Well, if you were bored by the foregoing, or if you read it with little attention or purely passively, you have indeed lost your time. Yet if you have solved, or honestly tried to solve, one or more of the three problems, and have had fun in doing so, and have carefully compared your effort with the printed solution, your time has been usefully spent: you had a good chance to improve your mind, to prepare yourself for problems which you may encounter in the future and which may be more serious.

You have a still better chance to improve your mind, that is, your ability to solve problems, if, having obtained the solution, or having read it with understanding, you look back at it, and think it over again: What was the most difficult, or the most prominent, point? What helped, what hampered you? What should you avoid, what should you seek the next

PÓLYA ON PUZZLES, FROM PAGE 3

time when you have to solve some similar problem? And to what kind of problem is this kind of solution applicable? Instant reflection on such questions may bring you the most profit.

Let us reflect a little on our solutions. They seem to be similar, they seem to have a common pattern. What is this pattern? In what does their similarity consist?

All three problems are "puzzles." Each problem presents several little scraps of information, which are disconcerting, embarrassing, "puzzling" because they seem to be so disconnected (they are not, in fact) and irrelevant (some are really irrelevant--it does not matter whether the purchase is made for a picnic or for Hallowe'en). The main achievement of the solution was, I think, to select a few items in that apparently disorderly heap of information and to combine them into a single major condition to satisfy which we have to solve a clear cut mathematical problem. This mathematical problem has several, but not too many, solutions. Now, those items of information which we have left aside in formulating our major condition constitute minor conditions which we use to eliminate some of the solutions satisfying the major condition. This elimination leaves just one solution. Yet this is the merit of our puzzles which are constructed according to the rules of the art: though we do not expect it when we read the puzzle the first time, the solution is unambiguous, uniquely determined.

There are serious problems in real life, for example in engineering design, in solving which we may attempt a somewhat similar procedure. An engineer has to design a new gadget. Now, such a new gadget has many demands to satisfy: sure working, customer appeal, ease of fabrication, low cost, durability, etc. Of this disconcerting mass of demands, the engineer tries to select a few principal ones which can be appropriately combined into a major condition, into a clear cut technical problem. This problem will have in general several solutions. The engineer surveys these solutions. Some of them will be eliminated by the remaining minor conditions, but still more than one solution will remain in most cases--real life is not so simple, it is not an artfully constructed puzzle. Still, occupation with puzzles may prepare us for more serious problems.

879-8447!

TELEPHONE DIAL 'CORRESPONDENCE' CONCEALS ATTRACTIVE OLD RIDDLE IN JINGLE FORM

One of the more interesting "correspondences" in our nonmathematical lives is between letters and numerals on the telephone dial. Thus--have you noticed? --A, B, and C correspond to 2, P, R, and S correspond to 7, everywhere in North America.

We live in an increasingly numeral age, and the letters on the dial see little service today. Such wasn't always the case. Once, big cities had named "exchanges," and the names and their initial letters were useful as a memory aid. Thus, the Editor's boyhood telephone number, HArbour 2065, was easier to recall than its all-numeral equivalent, 422065. The first two digits of distinctive, easily-remembered exchange names substituted for two of six (later seven) digits that had to be kept in mind.

Today, the use of the letters on the 'phone dial is all but forgotten . . . but they're still on the dials or by the buttons, and the letter-to-numeral correspondence can be the basis of an interesting, rather surprising, "fun" activity.

Thirty-three "telephone numbers"--they aren't really!--are given below. Copy the "correspondence" from a dial, then try this: substitute a letter for each digit, then read off the "message." If you succeed--which seems incredibly unlikely!--you'll be challenged by a rather attractive old riddle.

368-7565	596-3672	836-9686	752-9263	752-9332
556-4448	845-5273	256-3329	843-9752	933-3672
274-2636	683-6738	694-8473	727-2837	267-3736
738-3796	639-3894	368-4392	263-8675	927-3222
686-8784	392-5542	362-3359	483-3247	266-8687
226-9688	447-7272	369-3975	246-4366	663-5678
469-2685	325-5424	611-1101		

With each digit, in general, corresponding to three letters, the foregoing should permit an astronomical number of possibilities--the 225th power of 3--but wait! English is very redundant, the mind will seek patterns in the letter possibilities, and the riddle should all but jump out from the page.

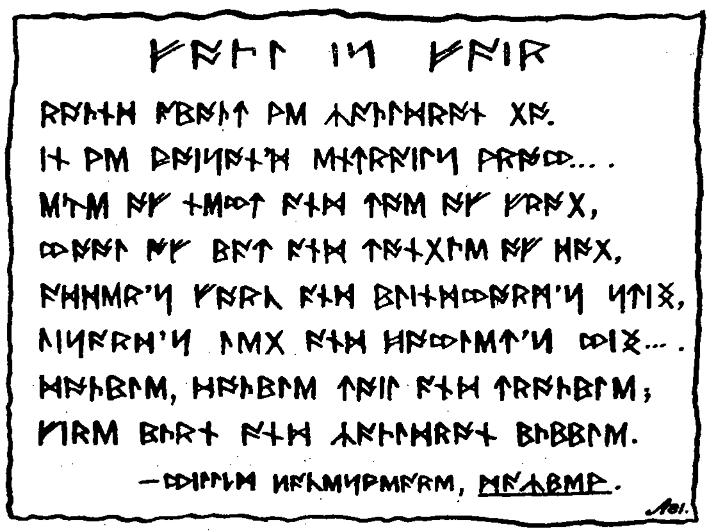
Try it!

"An anonymously written old jingle," we found it quoted by Martin Gardner in his Mathematical Puzzles (New York: Thomas Y. Crowell, 1961), p. 104.

The solution?

843-3687 565-5963 693-7336 875-6559 520-0687 424-2671!

H.D.A.

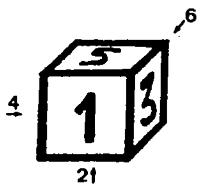


TREAT OR TRICK? Fall means Hallowe'en in North America, with Hallowe'en conjuring visions of witches, bubbling cauldrons, and improbable spells. Our Fall cryptogram, accordingly, looks to the more traditional "trick or treating"--and carries its message in a runic alphabet long associated with the writing down of spells! So try The Log's Hallowe'en special, but watch for two twists. The runic characters, Anglo-Saxon and earlier, do indeed constitute an alphabet, but (in our version) in two instances pairs of similar sounding consonants share a single sign. Also, our runic alphabet, as is traditional, extends to extra "letters," six for common consonant pairs (e.g., CH, SH, NG) and six for common vowel pairs (e.g., AE, EA, OA), not all characters occurring in our brief but classic "message." Have fun, but do not turn the page. The Runic Alphabet and comments on solution are given on page 6.

YEAR OF THE CUBE, FROM PAGE 1

face with the "cross." In working to this result, of course, care must be taken not to undo the efforts leading to Fig. 2.

To continue the "restoration" we need to consider specific situations and associated sequences of restoring "moves." To describe these moves, we will establish a numerical code. First, imagine the Cube to be a die, and the "cross" to have been placed on the "6" face of the die, opposite to face "1." (Recall: opposite faces on dice add to 7.) Suppose your "6" face to be bordered by faces "3," "5," "4," and "2," as illustrated at left. The next step in "restoring" the Cube will be to put a "cross" on face "1." To do this normally will call for two operations. First, edge cubelets must be in their proper places, even if this results in incorrect orientation--that is, the correct pair of colors, but each falling on the wrong face of the Cube. Then the edge cubelets must be correctly oriented. Figs. 4 and 5 relate to the moving of edge cubelets; Figs. 6 to 9 to corner cubelets. A Cube is viewed as presented in the Figure, Face "1" toward the viewer, Face "3" to the right, etc. Motions then can be coded by two digits: the first indicates the face on which one is working; the second indicates the number of quarter-turns counterclockwise that the face is to undergo. In Figs. 4 to 9, cubelets colored solidly interchange positions in the direction of the arrows; cubelets which are shaded change their orientation.



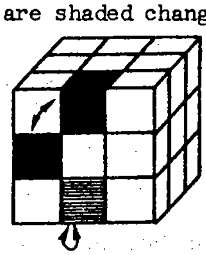


Fig. 4

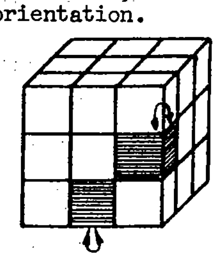


Fig. 5

To "work on" edge cubelets look to the situations depicted in Figs. 4 and 5. In Fig. 4, edge cubelets are made to interchange positions; also, incidentally, one of the other edge cubelets changes orientation. A sequence of "moves" which will accomplish this is:

11 23 13 33 11 31 21.

In Fig. 5, edge cubelets retain their positions but exactly two change their orientations. A sequence of moves which will accomplish this is:

23 31 21 33 11 33 13 31.

Use of these sequences, first to put edge cubelets in position, second to correct orientations, should bring the Cube to the stage where there is a "cross" on every face. Further efforts, therefore, are directed to corner cubelets. First, corner cubelets are worked into their proper places, regardless of orientation, using motions depicted in Figs. 6 to 8. Then, as necessary, corner cubelets are correctly oriented, by the motions associated with Fig. 9.

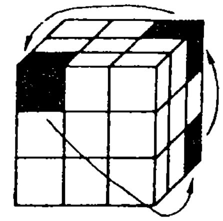


Fig. 6

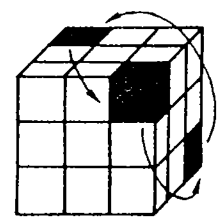


Fig. 7

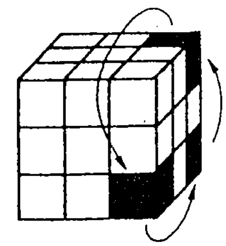


Fig. 8

The following six movements work corner cubelets into their proper places. As needed, they may be performed in any order.

Movement of Fig. 6, in the direction of the arrows:

52 11 22 13 52 11 22 13.

Movement of Fig. 6, in the direction opposite to the arrows:

11 22 13 52 11 22 13 52.

Movement of Fig. 7, in the direction of the arrows:

51 11 22 13 52 11 22 13 51.

Movement of Fig. 7, in the direction opposite to the arrows:

53 11 22 13 52 11 22 13 53.

Movement of Fig. 8, in the direction of the arrows:

12 52 11 22 13 52 11 22 11.

Movement of Fig. 8, in the direction opposite to the arrows:

13 22 13 52 11 22 13 52 12.

The following sequence serves to reorient corner cubelets (see Fig. 9). The reorientation which occurs is shown in Fig. 10.

51 63 53 33 63 31 13
33 61 31 51 61 53 11

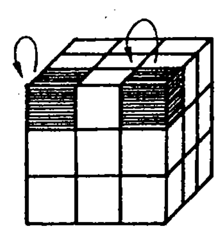


Fig. 9

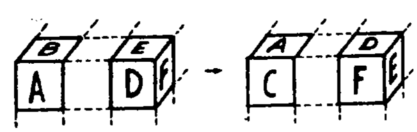


Fig. 10

Should anything go wrong in these later stages of Cube "restoration"--it can happen!--Math-Jeunes' advice (which The Log endorses) is to return rapidly and uncomplainingly to the stage depicted in Fig. 2.

Math-Jeunes' procedure, as here detailed, represents but one approach to "restoring" the Cube. The Scientific American article looks at some other possibilities. Reach for a cube, then begin! As you become proficient, you should recognize situations that allow for more direct methods of solution.

Good cubing! H.D.A.

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Hofstadter, Douglas R. "The Magic Cube's cubies are twiddled by cubists and solved by cubemeisters," in "Metamagical Themas," Scientific American, 244:3 (March 1981), pp. 20-39.

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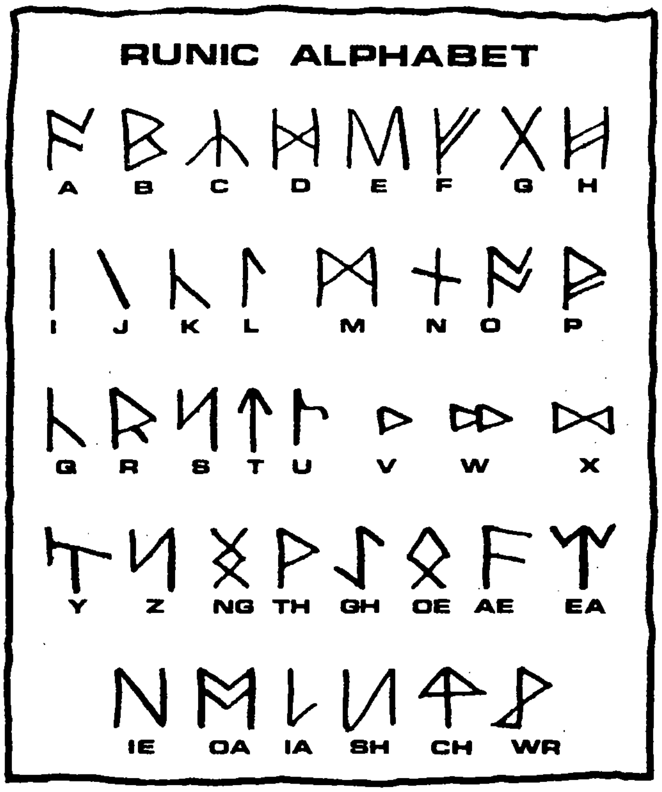
Majeunix, Le professeur P. I. (pseud.) "Le cube hongrois," Math-Jeunes, 3:8 (January-February 1981), pp. 45-47.

WHO SAID THAT?

EMMPW HMAFY LBSFN MABWV SIMRN MAHOZ
HXRJA XBHMM JMHJW YSNMA HOZHX RYZXG
MVSHN RBSNI HMLZF GEAXR SHX



HALLOWE'EN DECIPHERMENT! As used in our Hallowe'en cryptogram (page 4), a "modern" runic alphabet, based upon the Anglo-Saxon, is given above. Still earlier versions saw service in Germany and Scandanavia more than 1000 years ago. So, proceed to decipher! The encoded "spell" (don't try it!) is without doubt the most famous in English literature. If you've not met it in class, your English teacher can direct you to the source--and no doubt will commend you, The Log, and Mu Alpha Theta for broadened cultural interests!

dia**Log**ue with the editor

A box contains five hats, three black and two white. Three persons are blindfolded, then each is allowed to choose from the box one hat. The three persons then are seated in three chairs, the chairs arranged one behind the other. A sits in the front chair, B is behind A, and C is behind B. The three remain facing forward, and the blindfolds are removed. C can see the color of B's and A's hats, B can see only A's hat, and A cannot see any hats.

C promptly announces, "I don't know what color my hat is."

B then declares, "I don't know what color my hat is."

What color is A's hat?

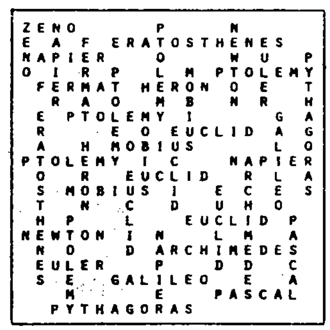
"Logical" puzzlers of this breed are perennial favorites. I recalled this one when I chanced upon a version of it in Moshe F. Rubinstein, Patterns of Problem Solving (Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1975), p. 87, while browsing in library stacks. The Rubinstein book is remarkable reading for its engineering systems thrust, its mature, diverse, demanding and worthwhile exploration of topics and techniques in problem solving, extending to some topics unusual to find in an elementary text (for example, non-linear programming and dynamic programming). So we recommend Rubinstein. And, while we're on the subject, we highly recommend as a library leisure activity browsing in mathematics stacks!

* * *

Competitions and questions in past Logs continue to bring in a worthwhile diversity of responses. A question on divisibility, "a679b is a five-digit number (in base ten) which is divisible by 72, determine a and b," was readily solved by Tom Bress, Riverdale High School, Fort Myers, FL, and by Song Tan, Southwest Miami Senior High School, Miami, FL. To be divisible by 72, a679b must be divisible by the (relatively-prime) divisors, 8 and 9. For the number to be divisible by 8, the three-digit number 79b must be divisible by 8 (why?), and b = 2. For a6792 to be divisible by 9, its digital root must be 9, so a + 6 = 9, and a = 3. As Song points out, 36792 = 72 x 511, which checks the solution.

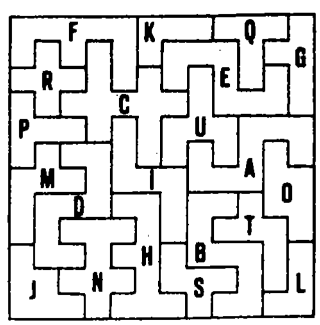
We're not sure we've heard the last word on our "'Crosswords' Challenge" (Winter Log), but some mighty impressive results have come to our desk. Below are the two best solutions to press time, a 260 for Mike Purcell, Danville High School, Danville, KY, and a 268 (!) for Terry Bridgman, Tuscola Senior High School, Waynesville, NC. Note the symmetry in Terry's solution. Do you see how, using this idea, Mike's might have been improved?

MIKE PURCELL, 260 POINTS



TERRY BRIDGMAN, 268 POINTS

The Lewis Carroll quotation in "pig-pen cipher" was solved and discussed by Scott Berkenblit, Valley Stream North (NY) High School chapter president, in a letter received as the Spring Log went to press. Log challenges, on occasion, go a bit further afield than chapter meetings, we now know. Our "Checkerboard Super Challenge" (Winter Log) was first solved by Dianne Barnes, a freshman at St. Vincent-St. Mary High School, Akron, OH--indeed, Dianne's solution (at left) is the only solution received to date. It was "sent in" by Deborah S. Patonai, the School's Mu Alpha Theta sponsor. The sponsor's letter ended on this, perhaps appropriate, concluding note:



How do you make the odd number SEVEN even?
Answer: Take away the S!