THE MATHEMATICAL LOG

Volume XXII, No. 2 January, 1978

THE GAME OF GOOGOL

I first encountered the game of Googol in the book New Mathematical Diversions from Scientific American by Martin Gardner, and I have been intrigued with it ever since. The game provides an interesting topic for several reasons. The optimum strategy for playing Googol is not immediately obvious, and its use greatly increases the chance of winning. The probabilities involved in the game are non-intuitive and somewhat surprising. Moreover, the game is accessible to analysis using only high school mathematics, the analysis is interesting, and it touches on several important mathematical concepts.

Let me describe the game: Googol is played by two persons, a writer and a guesser. The writer writes a different real number on each of n slips of paper without letting the guesser see them, and then hands the papers, face down, to the guesser. The guesser shuffles the papers (still face down), and then turns them over, one by one, until he stops and announces that the last number turned up is the maximum of the n numbers written. If he has guessed right, he wins the game; otherwise, he loses. Note that if the guesser passes by the maximum, he loses, since he is not allowed to go back to it. In particular, if he turns over every slip of paper, he loses, unless the last number is the maximum.

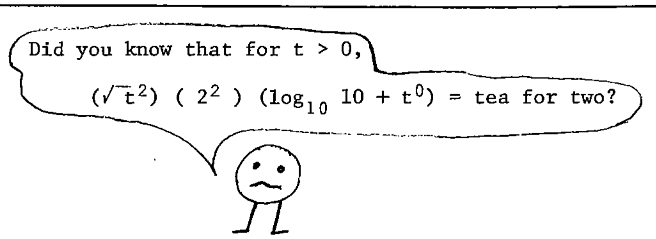
You may be able to keep the rules in mind more easily if you get an opponent and play a few games, using, say, $n = 10$ slips of paper. If the numbers written are all integers, the writer may want to use some very large integers. The game gets its name from the fairly large integer 10^{100}, which is called a "googol".

Now we want to examine several questions about the game of Googol. First, what is the best strategy for the guesser to use in picking the maximum? Second, what is the probability that he will win, using this best strategy? And finally, how does this probability change as the number n of slips of paper increases? Do you think the guesser would do much worse with 100 slips of paper instead of 10, or 10,000, instead of 100? We will study the game to answer these questions.

The analysis of the probability of winning at Googol requires only a few concepts from elementary probability, a little summation notation, and some juggling of inequalities. Here is a primer or refresher course in these "prerequisites".

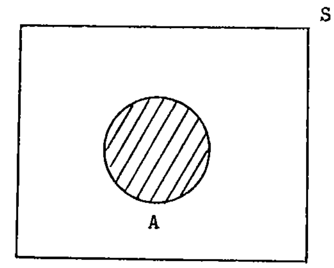
We will need three or five rules from probability. First, we will take the "equiprobable" interpretation of probability: If there are n equally likely outcomes of an experiment, and the event A occurs for k of these outcomes, then the probability of the event A is

(1) $P(A) = k/n$.



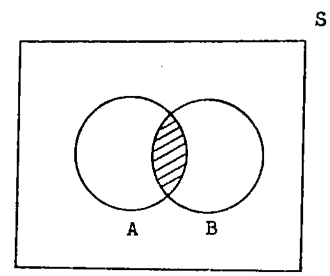
Now we will turn to the "dartboard" probability model to illustrate the other rules we'll need. Let S represent the whole dartboard and let A be a region of the board. The probability of hitting A with a "randomly thrown" dart will be

$$P(A) = \frac{m(A)}{m(S)}$$



where $m(A)$ is the area (or measure) of A and $m(S)$ is the area of S. Now suppose that we know that the dart has hit a region B, and we ask what is the (conditional) probability of hitting A given that the dart lands in B. The problem is really the same as finding the probability of hitting the intersection AB of the regions A and B on the smaller "dartboard" B. Clearly, the desired probability is

$$P(A/B) = \frac{m(AB)}{m(B)},$$



the area of the intersection AB divided by the area of B. Since

$$P(AB) = \frac{m(AB)}{m(S)} \text{ and}$$

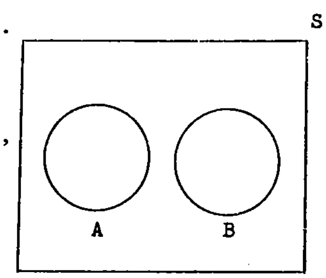
$$P(B) = \frac{m(B)}{m(S)},$$

substitution justifies the following definition: The conditional probability of the event A given the event B is

(2) $$P(A/B) = \frac{P(AB)}{P(B)} \quad (\text{if } P(B) \neq 0).$$

Hence, the probability of events A and B both occurring is

(2a) $$P(AB) = P(A/B)\cdot P(B).$$



Now suppose the regions A and B do not overlap. Thus, hitting A excludes the possibility of hitting B, and vice versa. Clearly, the probability of hitting A or hitting B is

$$P(A+B) = \frac{m(A+B)}{m(S)} = \frac{m(A)+m(B)}{m(S)} = P(A) + P(B).$$

This justifies the third rule: If A and B are mutually exclusive events, then the probability of A or B is

(3) $P(A+B) = P(A) + P(B)$.

If $A_1, \ldots, A_r$ are r mutually exclusive events, then the probability of A_1 or A_2 or ... or A_r is

(3a) $P(A_1 + \ldots + A_r) = P(A_1) + \ldots + P(A_r)$,

as an obvious generalization of (3). These three (or five) rules are all we'll need from probability theory.

It will be convenient for us to use the following summation notation:

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The official publication of the National High School and Junior College Mathematics Club, Mu Alpha Theta, which is sponsored by the Mathematical Association of America and the National Council of Teachers of Mathematics.
Address correspondence to: Mu Alpha Theta, 601 Elm Avenue, Room 423, The University of Oklahoma, Norman, Oklahoma 73019.

ANNOUNCEMENTS

The Mathematical Association of America now has available Volume I of the new Raymond W. Brink Selected Mathematical Papers series. This Volume is SELECTED PAPERS ON PRECALCULUS, and it would be a worthwhile addition for your mathematical library. It is an attractive book and is quite appropriate for Mu Alpha Theta students. Copies are available at a reduced price of $7.50 to Mu Alpha Theta members and sponsors. Also available at a reduced price of $10 per copy is Volume II in the series, SELECTED PAPERS ON CALCULUS. These books may be purchased from The Mathematical Association of America, 1225 Connecticut Ave., NW, Washington, DC 20036. A note from a sponsor should be included with each order certifying that those ordering are members of Mu Alpha Theta.

All correspondence concerning The Mathematical Log should be sent to: Dr. Betty Lichtenberg, Department of Mathematics Education, College of Education, University of South Florida, Tampa, FLorida 33620. If you're wondering what happened to something you mailed to Oklahoma, it was forwarded to Florida; and you'll hear soon.

The Third Annual South Carolina Mu Alpha Theta Convention will be held April 8, 1978, at the Baptist College in Charleston. Lynne Browning is the State President with Mrs. Ann Long and Miss Cheryl Coher the sponsors.

The Third Annual Convention of the Tennessee Assoc. Chapter of Mu Alpha Theta will be held March 10,11, at Bearden High School, Knoxville, TN. Sally Freschman, State Secretary, can be contacted for further information at: 1209 Redwood Avenue, Maryville, TN 37801.

ATTENTION - SPONSORS

If you are planning to attend the annual convention of Mu Alpha Theta in San Diego, April 12-15, 1978, include the following on your agenda.

(a) Breakfast for Sponsors on Friday, April 14, at 7:30 a.m. at the Executive Hotel. Provides a chance to visit informally with other sponsors and some members of the Governing Council. Price is $4.25 plus tax and gratuity. Reservations (pay at the meeting) should be sent to Harold Huneke, Math Dept., University of Oklahoma, Norman, OK 73019, by April 10.

(b) Section meeting on Mathematics Clubs sponsored by Mu Alpha Theta, Saturday, April 15, at 10:30 a.m.

1978 NATIONAL CONVENTION

Plan now to attend the 1978 convention, August 6-9, at Stevens Point, Wisconsin. Your chapter should already have received information on the meeting, but in case you have not, please write to Mr. Robert Meyer, Tomahawk High School, Tomahawk, WI 54487.

Over 200 chapters have expressed an interest, so it should be a great convention. Plan now to have a team in the Math Bowl competition, or some of your members may wish to present a student paper.

STUDENT DELEGATE ASSEMBLY AT NATIONAL CONVENTION

The Governing Council of Mu Alpha Theta, at its August, 1977, meeting, voted to establish a Student Delegate Assembly at each national convention. To implement this, each chapter which plans to attend the convention next August should elect a delegate and send their name and home address to the national office of Mu Alpha Theta, 601 Elm Avenue, Room 423, University of Oklahoma, Norman, OK 73019.

The Student Delegate Assembly will have the responsibility of generating ideas and recommendations to improve Mu Alpha Theta. Details of the organization of the Assembly will be sent to the elected delegates at a later date.

APOLOGIES FROM THE EDITOR

Dr. Louise Grindstein's name was inadvertently omitted from the lead article in the fall issue. She is a professor at Kingsborough Community College of the City University of New York. Watch for more about women and mathematics.

LETTERS TO THE EDITOR

Dear Mathematical Log:

Greetings from a new chapter! We noticed your plea for some ABSOLUTELY SILLY 's, so here are a few of our more "punny" ones:

Cold cuts. . . . . .Transformations in polar coordinates.

In fear of the Huns. . . . . Hundred.

If you want to lose weight. . . . . Trinomials.

Backward billboard. . . . . Inverse sine.

A pessimist. . . . . .Negative one.

What people do when they both sign for a loan. . .Cosine.

And how about a new column: QUICK "QUOTES"

"Steel belted radicals"

"Power to the exponent"

"Square roots planted in a field"

"Temperatures in the n^{th} degree"

We better end here before things get out of hand.

Douglas Mooers
Bishop Garcia High School
Santa Barbara CA 93110

Dear Editor:

Your interesting computation on the last page of the last issue of the Mathematical Log requested a proof. I'm sure many have or will respond, but probably the simplest proof is just recognizing the problem as a factoring problem involving the difference of two perfect squares. The proof could then go as follows:

Prove: $\underbrace{(55\ldots56)}_{n\text{ digits}}{}^2 - \underbrace{(44\ldots45)}_{n\text{ digits}}{}^2 = \underbrace{111\ldots11}_{2n\text{ digits}}$

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(4) $\sum_{m=k}^{n} a_m = a_k + a_{k+1} + \dots + a_n$.

Just remember that the upper case Greek letter sigma, "Σ", means to sum or add, and the bottom and top indices k and n tell where to start and where to stop. (It is sometimes convenient to let the "empty sum"

$$\sum_{m=k}^{k-1} a_m = 0.)$$

The rules of inequality we'll use are the standard rules you already know: The trichotomy rule

(5) for any real numbers a and b, exactly one of $a < b$, $a = b$, or $a > b$ occurs;

the transitive rule

(6) $a < b$ and $b < c$ implies $a < c$;

the addition rule

(7) $a < b$ iffi * $a + c < b + c$;

the multiplication rule

(8) if $c > 0$, then $a < b$ iffi $ac < bc$;

the inverse rule (This rule follows from (5) and (8).)

(9) $0 < a < b$ implies $0 < 1/b < 1/a$;

and the absolute value (or triangle inequality) rule

(10) $|a + b| \le |a| + |b|$.

We will also use similar rules involving $\le$. All of the inequalities that follow can be derived from the above rules, but space limitation will usually preclude a reference to the pertinent rule. As you read, you should verify the validity at each step in the argument; it may require some thought, but should not be too difficult. If you get stuck on a concept, read on at least to the end of the paragraph. Often this will clarify the situation. If not, review the pertinent rules and definitions. If you still cannot resolve the problem, accept the concept on faith and continue reading. Hopefully, you will be able to fill in any gaps in your understanding after you have read the whole presentation.

The analysis of Googol begins with a qualitative decision on optimum strategy. Suppose you and I are playing, and I am the guesser. You have just handed me n slips of paper, and I have shuffled them, face down. If I randomly choose the maximum, I have one chance in n of winning. (This is great if n = 1 or 2, but not so good for n = 100 or 1000, or even for n = 3, as we shall see.) The only way I can do better is to obtain some information about the numbers listed by looking at some of them without guessing and then to use this information in making my decision. Since there is no reason for me to expect a pattern in the numbers you wrote, the only reasonable decision rule to use <u>when guessing</u> is to choose a number as the maximum if it is the largest yet seen, and to reject it otherwise. Hence, the optimum strategy is as follows: I will look at the first k papers without guessing, then I will choose the first number which is larger than the largest among the first k.

To optimize my strategy, I must find what value of k gives me the best chance of winning. Here we must use a quantitative study, which is best started with a few definitions:

<u>Definition 1.</u> p(n,k) is the probability that the guesser wins, using n slips of paper and looking at k slips without guessing

<u>Definition 2.</u> For each positive integer n,

$$p(n) = \max\{p(n,k) : k = 0, 1, \dots, n-1\}.$$

Thus, p(n) is the largest of the probabilities p(n,0), p(n,1), ... , p(n,n-1), and hence is the best possible probability of winning with n slips of paper.

* "iffi" means "if and only if".

<u>Definition 3.</u> For each positive integer n,

$$k(n) = \min\{k : k=0, 1, \dots, n-1 \text{ such that } p(n,k) = p(n)\}.$$

(Actually, if $n \neq 2$, there is only one k giving p(n,k)= p(n), so it is unnecessary to take the minimum k such that p(n) = p(n,k). We use the above definition so we won't have to prove that there is only one such k when $n \neq 2$.)

Observe that

(11) $p(n,0) = p(n,n-1) = \frac{1}{n}$,

since k = 0 means the guesser arbitrarily selects the first number as maximum, and k = n-1 means he arbitrarily selects the last. Let's evaluate p(n,k) and k(n) for small n and k:

$$(12)\quad \begin{cases} p(1) = p(1,0) = 1; \quad k(1) = 0. \\ p(2) = p(2,0) = p(2,1) = \tfrac{1}{2}; \quad k(2) = 0. \\ p(3,0) = p(3,2) = 1/3, \; p(3) = p(3,1) = \tfrac{1}{2}; \\ \quad k(3) = 1. \end{cases}$$

All of these values except p(3) = p(3,1) are obtained from (11). To get p(3,1), we suppose that the numbers $a < b < c$ are written. After shuffling, each of the six orderings abc, acb, bac, bca, cab, and cba are equally likely, and the guesser wins on the three arrangements acb, bac, and bca. Thus, p(3,1) = 3/6 = 1/2 by the equiprobable rule (1). Using the best strategy (k(3) = 1) increases the chance of success from 1/3 to 1/2!

Next on the agenda is the calculation of a general formula for p(n,k). Suppose that after shuffling, the numbers are arranged in the order

$$a_1, \dots, a_k, a_{k+1}, \dots, a_n.$$

Each of the n numbers a_i has the same chance of being the largest, so the probability that a_{m+1} is the maximum is

(13) $P(a_{m+1} \text{ is max}) = P(A_m) = 1/n$

for each m = 0, 1, ... , n-1. As in (13), let A_m denote the event that a_{m+1} is the maximum. Let W denote the event that the guesser wins. The probability of winning is 0 if the maximum is among $a_1, \dots, a_k$, so

(14a) $P(W/A_m) = 0$ for $m = 0, 1, \dots, k-1$.

Suppose now that a_{m+1} is the maximum and $m \ge k$. Consider the array

$$a_1, \dots, a_k, a_{k+1}, \dots, a_m, a_{m+1}, \dots, a_n.$$

Let a_i be the largest of the numbers $a_1, \dots, a_m$. If $i > k$, the guesser will choose a_i or another a_j larger than $a_1, \dots, a_k$ instead of the true maximum a_{m+1} and lose, since a_i is larger than $a_1, \dots, a_k$ and comes before a_{m+1}. But if $i \le k$, then a_{m+1} is the first number larger than all of $a_1, \dots, a_k$, so the guesser selects a_{m+1} and wins. Thus, when $m \ge k \ge 1$, the probability of winning given that a_{m+1} is the maximum is simply

(14b) $P(W/A_m) = k/m$ for $m = k, k+1, \dots, n-1$ $(k > 0)$,

since this is the probability that the maximum a_i of the first m terms occurs among the first k terms. (We again use the equiprobable rule (1), since there are m equally likely choices $a_1, \dots, a_m$ for the maximum a_i, and k of these choices give a win.)

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Using (2a), we see that the probability that the guesser wins (event W) and that the maximum is a_{m+1} (event A_m) is $P(WA_m) = P(W/A_m)\cdot P(A_m)$, so it follows from (13) and (14ab) that

$$(15)\quad P(WA_m) = \begin{cases} 0 & \text{if } m = 0, 1, \ldots, k-1 \\ \frac{k}{mn} & \text{if } m = k, k+1, \ldots, n-1 \end{cases}.$$

Now the event W of winning is the "logical sum"

$$W = WA_k + WA_{k+1} + \ldots + WA_{n-1}$$

of the mutually exclusive events WA_m (m=k,...,n-1) of winning with the maximum a_{m+1}. Hence

$$p(n,k) = P(W) = P(WA_k) + \ldots + P(WA_{n-1})$$

by probability rule (3a). Substituting (15) into the above and factoring out the constant k/n gives

$$(16)\quad p(n,k) = \frac{k}{n}\sum_{m=k}^{n-1}\frac{1}{m} \quad (k > 0,\ n > 1),\qquad p(n,0) = 1/n.$$

as the probability that the guesser wins by looking at k slips before starting to guess. The special case for k = 0 can be avoided by writing

$$(16a)\quad p(n,k) = \frac{1}{n} + \frac{k}{n}\sum_{m=k+1}^{n-1}\frac{1}{m} \quad (k = 0, 1, \ldots, n-1).$$

(Note that if k = n-1 in (16a), the sum is 0, since it is the "empty sum" in that case.)

Our next job is to find the values of k which make p(n,k) as large as possible. For this purpose we want to see when p(n,k) is less than, equal to, or greater than p(n,k+1). To save space, I will use the symbol " $\lesseqgtr$ " to represent the three possibilities, since the argument will be the same, whether you always read the "<", always read the " = ", or always read the " > ". Substituting from (16) and (16a),

$$p(n,k) \lesseqgtr p(n,k+1)$$

iffi

$$\frac{1}{n} + \frac{k}{n}\sum_{m=k+1}^{n-1}\frac{1}{m} \lesseqgtr \frac{k+1}{n}\sum_{m=k+1}^{n-1}\frac{1}{m} = \frac{k}{n}\sum_{m=k+1}^{n-1}\frac{1}{m} + \frac{1}{n}\sum_{m=k+1}^{n-1}\frac{1}{m}$$

iffi

$$\frac{1}{n} \lesseqgtr \frac{1}{n}\sum_{m=k+1}^{n-1}\frac{1}{m}$$

iffi

$$1 \lesseqgtr \sum_{m=k+1}^{n-1}\frac{1}{m}.$$

Thus, for n > 1, and k = 0, 1, ... , n-2,

$$(17)\quad p(n,k) \lesseqgtr p(n,k+1) \quad \text{iffi} \quad 1 \lesseqgtr \sum_{m=k+1}^{n-1}\frac{1}{m}.$$

Now the sum

$$\sum_{m=k+1}^{n-1}\frac{1}{m} = \frac{1}{k+1} + \frac{1}{k+2} + \ldots + \frac{1}{n-1}$$

is equal to the sum

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1}$$

when k = 0, and "lops off" terms on the left in the latter sum as k increases. If n > 2, the sum starts out greater than 1 for k = 0, then the sum decreases as k increases, until finally (for k = n-2) the sum is $\frac{1}{n-1}$, which is less than 1. Thus if k* is the smallest k such that the sum is ≤ 1, we see that the "<" holds in (17) when k < k*, " > " or " = " holds in (17) when k = k*, and " > " holds when k > k*. Hence,

$$p(n,k) < p(n,k+1) \quad \text{if} \quad k < k^*,$$
$$p(n,k) \geq p(n,k+1) \quad \text{if} \quad k = k^*,$$
$$p(n,k) > p(n,k+1) \quad \text{if} \quad k > k^*,$$

so p(n,k) increases until k = k*, where p(n,k*) = p(n) is maximum, and then p(n,k) decreases. Thus the smallest k (k = k* = k(n)) giving p(n,k) = p(n) occurs for p(n,k-1) < p(n,k) and p(n,k) ≥ p(n,k+1). Hence, using (17) "as is" and with k replaced by k-1, we have for n ≥ 3*,

$$p(n,k) = p(n) \text{ and } k = k(n)$$

$$(18)\quad \text{iffi} \quad \sum_{m=k+1}^{n-1}\frac{1}{m} \leq 1 < \sum_{m=k}^{n-1}\frac{1}{m}.$$

We now know how to play Googol with the optimum strategy! Simply add the reciprocals "backwards", starting with $\frac{1}{n-1}$, until the sum $\frac{1}{n-1} + \frac{1}{n-2} + \ldots + \frac{1}{k}$ first, exceeds one; this value of k is k(n). Then play the game by observing the maximum of the first k(n) numbers without guessing, and then choosing the next number which exceeds this maximum. For n = 10, we note that

$$\frac{1}{9} + \frac{1}{8} + \frac{1}{7} + \frac{1}{6} + \frac{1}{5} + \frac{1}{4} < 1,$$

but

$$\frac{1}{9} + \frac{1}{8} + \frac{1}{7} + \frac{1}{6} + \frac{1}{5} + \frac{1}{4} + \frac{1}{3} > 1,$$

so

$$k(10) = 3 \text{ and } p(10) = \frac{3}{10}\sum_{m=3}^{9}\frac{1}{m} \doteq 0.398690.$$

* (18) works for n = 1 or 2 if we interpret 1/0 > 1.

A study of the behavior of P(n) for large n yields even more interesting mathematical results. It turns out that when playing Googol with n larger than 10, the optimum value of k can be estimated fairly well by n/e, or even n/3 if n is not too large.

Happy Googoling!

William P. Wardlaw
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Annapolis, MD 21402

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Proof:

$$(\underbrace{55\ldots56}_{n\text{ digits}})^2 - (\underbrace{44\ldots45}_{n\text{ digits}})^2 = (\underbrace{55\ldots56}_{n\text{ digits}} + \underbrace{44\ldots45}_{n\text{ digits}})(\underbrace{55\ldots56}_{n\text{ digits}} - \underbrace{44\ldots45}_{n\text{ digits}})$$

$$= (\underbrace{100\ldots0001}_{n+1\text{ digits}})(\underbrace{111\ldots11}_{n\text{ digits}})$$

$$= (1\underbrace{00\ldots00}_{n\text{ zeros}}+1)(\underbrace{111\ldots11}_{n\text{ digits}})$$

$$= \underbrace{11\ldots11}_{n\text{ ones}}\ \underbrace{00\ldots00}_{n\text{ zeros}} + \underbrace{111\ldots11}_{n\text{ digits}}$$

$$= \underbrace{1111\ldots1111}_{2n\text{ digits}}$$

Please continue publishing interesting problems such as this.

Our local chapter of Mu Alpha Theta is planning a Math Game Fair on December 9th, and will sponsor its annual Elementary School Mathematics Contest on March 21, 1978. Last year 10 schools participated in this contest. Please keep us posted about the convention.

(Dr.) Betty L. Baker, sponsor
Hubbard High School
Chicago, IL 60629