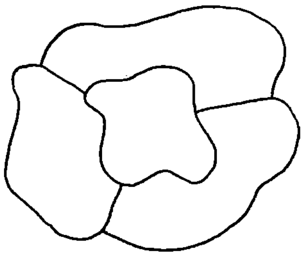
THE MATHEMATICAL LOG

Volume XXI, No. 1 September, 1976

STOP THE PRESSES-The four-color problem is now solved.

The following map requires four colors if it is to be colored so that no two countries having a common boundary are the same color.



Mapmakers have never found a map, no matter how complicated, that requires more than four colors.

About 125 years ago, mathematicians became interested in this; and since then, they have been trying to prove that four colors would be sufficient for any map in a plane.

Professors Kenneth Appel and Wolfgang Haken at the University of Illinois, have solved this famous problem-they have proved that no map in a plane requires more than four colors.

An interesting feature of their proof is that it involved extensive use of a computer to test what they had previously proved were all of the possibilities. And because of these two mathematicians, a lot of mathematics books are sure going to have to be revised. (See Time Magazine, Sept. 20, 1976. pages 87-88.)

KHAJEH NASSEER TOOSI

For many great men, even in the twelfth century, recreational mathematics had its importance. Among Kahjeh Nasseer Toosi's mathematical works one finds a magic poem. He has used properties of base 2 in composing a magic poem. (Fig. 1) In order to avoid criticism of fanatics, he has selected the letters of a prayer. Thus many people believed that the magic had been obtained from those letters.

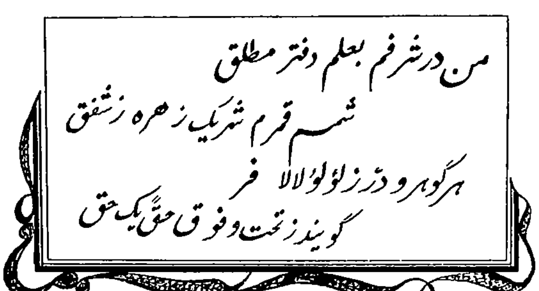


Fig. 1

1. A poem. We have composed a poem with properties of Toosi's poem. It may sound slightly humorous, but imposing so many conditions we had to choose these words:

Honored am I by a knowledge aura,
Nadir to Zenith, Pearls of Sun and Moon
By Mercury, Mars, Jupiter, Uranus, Neptune;
Bathing with Zodiac, Venus, and Aurora.

Now we ask a friend to select a letter from the table below:

Q	K	F	L	J	Y	P	M	V	G	Z	D	C	B	T	N
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

For example, let M be chosen. Since M is in the first line of our poem, we choose 1. The letter M is also in the second line. So for the second line we also choose 1. Since M is also in the third line, we have another 1 for the third line. But there is no M in the fourth line; thus 0 corresponds to the fourth line. This way we have $(0111)_2$ in the base two. Changing $(0111)_2$ to the base ten, we get $(0111)_2 = 4 + 2 + 1 = 7$. In the table above 7, we find M. The poem has been composed to have this property for all letters of the table. Of course, one finds that Q corresponds to zero.

2. Magic in a poem: One may choose a four-line poem and change it to a magic poem. For example, we select a nursery rhyme:

Ride a cock-horse to Banbury Cross,
To see a fine lady upon a white horse;
Rings on her fingers and bells on her toes,
She shall have music wherever she goes.

We examine every letter and find the number corresponding to it. For example, B is in the first and the third lines. Thus to B corresponds 5. Most vowels occur in all four lines. So we ignore them unless we need them. Examining this poem we obtain:

$1 \leftrightarrow \{K\}$, $2 \leftrightarrow \{P\}$, $3 \leftrightarrow \{Y\}$, $4 \leftrightarrow \phi$, $5 \leftrightarrow \{B\}$

$6 \leftrightarrow \{F\}$, $7 \leftrightarrow \{D,N,T\}$, $8 \leftrightarrow \{M,V\}$, $9 \leftrightarrow \{C\}$ $10 \leftrightarrow \{W\}$

$11 \leftrightarrow \{U\}$, $12 \leftrightarrow \{G\}$, $13 \leftrightarrow \phi$, $14 \leftrightarrow \{L\}$, $15 \leftrightarrow \{H,R,S\}$

Note that to each number corresponds a set of letters. Here, to 4 or 13 corresponds the empty set. Thus for certain numbers we have one choice, but for others more than one. Here we may choose:

J	K	P	Y	B	F	D	M	C	W	U	G	L	R
0	1	2	3	5	6	7	8	9	10	11	12	14	15

One observes that 4 and 13 are missing. But we can still play the game.

Now we shall give another example. We select a Fitzgerald's translation of Khayyam's poem.

Were it not Folly, Spider-like to spin
The Thread of present Life away to win --
What? for ourselves, who know not if we shall
Breathe out the very Breath we now breathe in!

This poem has such repetition of letters that the best we can do is to obtain the following table:

D	K	F	B	Y	U	H	R
3	5	7	8	11	12	14	15

Indeed, one can play the game. This is only to demonstrate that any four-line poem can be transformed into a magic one. One may choose a poem which has more than four lines. We shall not go into that.

(continued on pg. 4, col. 2)

The official publication of the National High School and Junior College Mathematics Club, Mu Alpha Theta, which is sponsored by the Mathematical Association of America and the National Council of Teachers of Mathematics. Address correspondence to: Mu Alpha Theta, 601 Elm Avenue, Room 423, The University of Oklahoma, Norman, Oklahoma 73019.

SECRETARY'S CORNER

Last year was the best year in the history of Mu Alpha Theta. 18,520 students joined and 84 chapters were chartered.

New officers and governors you elected last year are:

President-Elect	Robert Kalin
Governor Region I	James Woolum
Governor Region II	Kathryn Hinsenbrock
Secretary-Treasurer	Harold Huneke

A revised version of The Handbook for Sponsors is being prepared and will be mailed out to all chapters this fall.

Mu Alpha Theta pins, shoulder patches, banners and publications are available from the national office. The prices are the same as last year. If you need order forms, please let us know.

Six state and six regional meetings were held last year. Plan one for your state or region this year.

ANNOUNCEMENTS

The annual High School Contest which Mu Alpha Theta co-sponsors is scheduled for March 8, 1977. Last year 356,000 students took the examination. A list of regional coordinators is included in this mailing of The Log. Contact them for information. Order forms for recent examinations and solution keys are available on request from Dr. Walter E. Mientka; 917 Oldfather Hall; University of Nebraska; Lincoln, Nebraska 69588

A breakfast for Mu Alpha Theta sponsors is scheduled for April 22, 1977, at 7:30 a.m. at the Cincinnati meeting of NCTM. Contact the secretary, Dr. Huneke, for information and reservations.

THE NATIONAL CONVENTION WILL BE HELD AUGUST 7-10, 1977, AT LORAS COLLEGE, DUBUQUE, IOWA.

Start planning now to attend, compete in the Math Bowl, present a student paper, etc.

You will receive a mailing with detailed information during November. Joyce Hubka, Wahlert High School, 2005 Kane Street, Dubuque, Iowa 52001, is serving as general chairman.

MATH FAIR PAPERS

The papers for the first booklet of Math Fair Papers are now being edited for publication. This is a continuing project and papers are now being accepted for the second booklet. When a paper is submitted, the following information should be given:

1. Title of paper
2. Author's name, address and phone number
3. School's name and address
4. Date, place and name of Math Fair competition at which the award was won, and the nature of the award. If no Math Fair is held in your region, a recommendation from a sponsoring teacher will suffice.
5. If there was a sponsor, his/her name and address.

Mail four copies to: Harry Ruderman, 2624 Davidson Ave., Bronx, New York 10468

Enclose a self-addressed envelope for returning the paper if it is not selected for publication.

TEXAS INSTRUMENTS

Prices in the calculator market are dropping rapidly and thus discounts allowed are smaller.

The list and discount prices as of August 1, 1976, for the two most popular calculators are:

SR 50A	List Price \$59.95,	Discount Price \$50.95
SR 51A	List Price \$79.95,	Discount Price \$71.50

The discount price includes the mailing cost and a one year warranty. If you wish to purchase any calculator please send the order to our office, National High School Mathematics Club, 601 Elm Ave., Rm 423, Norman, OK 73019. The check should be made out to "National High School Mathematics Club".

You may wish to delay any purchases since we are informed that the SR-50A and SR-51A will soon be discontinued and replaced by the SR 40 which will retail for \$39.95.

CORRECTION

Kurt Bomar's name was spelled incorrectly in the last issue of The Mathematical Log. Sorry about that, Curt.

ABSOLUTELY SILLY

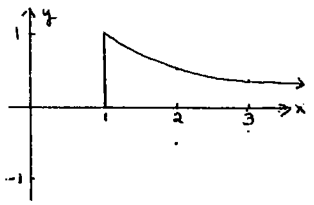
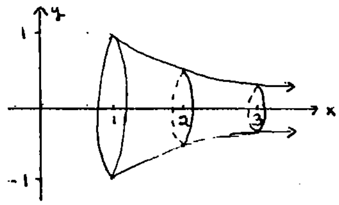
What did your teacher tell Calvin, the maintenance man, when the air conditioner broke? ....Calculus

What will you do if you lose your lunch money?....binomial

(A ten-sided polygon is a decagon and a twelve-sided polygon is a dodecagon.)
Do you believe that a ten-sided pizza is a ..dough-decagon
and a twelve-sided pizza is a ....dough-dodecagon
and (worse than that) a twelve-sided pizza for an extinct bird is a ....dodo dough-dodecagon
and (horrible!) after that's eaten, it's a............... dodo dough-dodecagon gone?

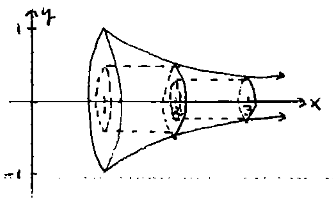
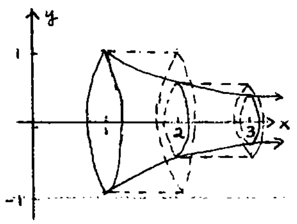
AN INTERESTING SOLID

In the calculus one deals with the problems of finding the surface areas and volumes of non-standard solids. One solid which I find interesting is obtained by revolving the region bounded by $y = \frac{1}{x}$, $y = 0$ and $x = 1$ about the x axis. The solid extends infinitely to the right and each cross section perpendicular to the x axis is a circular region.

Suppose we try to answer the following questions: (a) What is the surface area of this solid? (b) What is the volume of this solid? We will be able to obtain partial answers to these questions without the calculus.

First we approximate the solid by inscribing cylinders of height one and with a base of radius equal to the height of $y = \frac{1}{x}$ at $x = 2, 3, \ldots$ as indicated in the figure on the left. Let us also approximate the solid by circumscribing cylinders of height one and with a base of radius equal to the height of the curve at $x = 1, 2, \ldots$ as indicated in the sketch on the right.

The following inequalities are obvious: (a) The sum of the lateral areas of the inscribed cylinders is less than the surface area of the solid. (b) The volume of the solid is less than the sum of the volumes of the circumscribed cylinders.

Now the surface area of a cylinder equals $2\pi rh$ so we have the following infinite sum for the sum of the lateral areas of the inscribed cylinders,

$$2\pi\left(\frac{1}{2}\right)(1)+2\pi\left(\frac{1}{3}\right)(1)+2\pi\left(\frac{1}{4}\right)(1)+\ldots=2\pi\left[\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\right]$$

If we can decide on how to calculate this infinite sum we will have some idea about the surface area of the original solid. (We will delete the factor of 2π in the following discussion.)

Mathematicians agree to handle infinite sums in a rather natural way by studying a sequence of approximations usually called partial sums. As a first approximation S_1, we take the first term, in this case $S_1 = \frac{1}{2}$, as a second approximation S_2, we take the sum of the first two terms, in this case $S_2 = \frac{1}{2} + \frac{1}{3}$, and in general the nth approximation S_n is the sum of the first n terms, in this case $S_n = \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n+1}$. The problem of studying the infinite sum is thus reduced to a problem of studying the sequence $S_1, S_2, S_3, \ldots S_n, \ldots$ of approximating sums.

If the terms of this sequence get "close to" some number S as n increases, we say the limit of the sequence is S (the sum of the series is S) and write $\lim_{n\to\infty} S_n = S$. Later in mathematics you will make the meaning of "close to" and this limit more precise.

I now plan to convince you that as n becomes infinite so does S_n which implies that the sum of the surface areas of the inscribed cylinders is infinite which in turn implies the surface area of the original solid is infinite.

Instead of considering all of the terms in the sequence $S_1, S_2, \ldots, S_n, \ldots$ we will consider only a subsequence. My choice of the subsequences is based on the fact that each term in the subsequences is at least $\frac{1}{2}$ greater than its predecessor. I now exhibit a subsequence with this property and you should observe the pattern.

$S_1 = \frac{1}{2}$.

$S_3 = S_{2^2-1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{2} + 2\left(\frac{1}{4}\right) = 2\left(\frac{1}{2}\right)$.

So, $S_{2^2-1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 2\left(\frac{1}{2}\right)$.

$S_7 = S_{2^3-1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} >$
$\frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 3\left(\frac{1}{2}\right)$.

So, $S_{2^3-1} > 3\left(\frac{1}{2}\right)$ and in general, for $n > 1$, $S_{2^n-1} > n\left(\frac{1}{2}\right)$.

Note that writing 3 as 2^2-1 and 7 as 2^3-1 helps us in recognizing the pattern.

Now as $n \to \infty$, $n(\frac{1}{2}) \to \infty$ so the subsequence $S_1, S_3, S_7, S_{15}, \ldots, S_{2^n-1}, \ldots$ becomes infinite, that is, unbounded. The reader should try to present an argument that this implies that the original sequence $S_1, S_2, \ldots, S_n \ldots$ must also become infinite.

I claim that I have established that the surface area of the original solid is infinite. At this stage your reaction may be that since the solid extends infinitely to the right then it is not surprising that the surface area is infinite. I would now like to convince you that the volume is finite and thus the solid in question has infinite surface area but finite volume and this is rather intriguing.

First we need another result. Suppose you divide 1 by $(1-x)$ to obtain the following: $\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^{n-1} + \ldots$ Are the two sides actually equal? For $x = 2$, $\frac{1}{1-2} = -1$ while the right side becomes $1 + 2 + 2^2 + \ldots + 2^{n-1} + \ldots$ which obviously does not equal -1. For $x = \frac{1}{2}$, $\frac{1}{1-x} = \frac{1}{1-\frac{1}{2}} = 2$ and the right side becomes $1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} + \ldots$ Could this infinite sum equal 2 in some sense? To answer this we need to study the sequence of partial sums $S_1, S_2, \ldots, S_n, \ldots$ In this case it is possible to obtain a simple expression for S_n which enables us to argue that as $n \to \infty$, $S_n \to 2$. Now $S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}}$ and we can multiply both sides by $\frac{1}{2}$ to obtain $\frac{1}{2}S_n = \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n}$ and on subtracting we have $\frac{1}{2}S_n = 1 - \frac{1}{2^n}$ or $S_n = 2 - \frac{1}{2^{n-1}}$. Now as n gets larger and larger, $\frac{1}{2^{n-1}}$ gets smaller and smaller so $\lim_{n\to\infty}\left(2 - \frac{1}{2^{n-1}}\right) = 2$ and the series $1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} + \ldots$ converges to 2. (This series is a special case of a geometric series $a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$ which you may have already studied.)

(continued on pg. 4, col. 1)

(An Interesting Solid-cont.)

With this background we consider the sum of the volumes of the circumscribed cylinders. Recalling that the volume of a cylinder is $\pi r^2 h$, we have for the sum of the volumes of the circumscribed cylinders,

$$\pi(1)^2(1) + \pi(\tfrac{1}{2})^2(1) + \pi\left(\frac{1}{3}\right)^2(1) + \ldots = \pi\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right]$$

If I can show the series $\frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{n^2} + \ldots$ converges to a finite sum then you would have to agree that the original solid has a finite volume.

As in the series for the surface area we consider a subsequence of the partial sums. You should note the pattern used in obtaining the inequalities.

$$S_1 = 1,\ S_3 = 1 + \frac{1}{2^2} + \frac{1}{3^2} < 1 + \frac{1}{2^2} + \frac{1}{2^2} = 1 + 2\left(\frac{1}{2^2}\right) = 1 + \frac{1}{2},$$

$$S_7 = S_{2^3-1} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} < 1 + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2} = 1 + 2\left(\frac{1}{2^2}\right) + 4\left(\frac{1}{4^2}\right) = 1 + \frac{1}{2} + \frac{1}{2^2}.$$

In general, $S_{2^n-1} < 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}}$. But the series $1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} + \ldots$ converges to 2 and each partial sum is less than two so $S_{2^n-1} < 2$ for all n.

Now, $S_1, S_3, S_7, \ldots, S_{2^n-1}, \ldots$ is an increasing sequence (always adding positive terms in the partial sums) with each term less than 2, and thus it is intuitively clear that as $n\to\infty$, S_{2^n-1} converges to some value less than 2. (To justify this requires the one property of the reals which is not shared by the rationals, that is, every increasing sequence of reals which is bounded above must have a limit which is a real number.) The reader might try to construct an argument that the original sequence $S_1, S_2, \ldots, S_n, \ldots$ also converges to the same value.

Summarizing, the series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{2^{n-1}} + \ldots$ converges to a value less than two so the sum of the volumes of the circumscribed cylinders is less than 2π which implies the volume of the original solid is finite, actually less than 2π cubic units.

If you think of the solid as hollow then this seems to imply that you can fill it with paint (volume less than 2π cubic units) but you cannot paint it (surface infinite). The Mathematical Association of America has produced a 10 minute film called "Infinite Acres" which deals with this problem. A college or university in your area may have a copy of the film. Possibly you could discuss this problem and show the film as a club program. I think it is a "fun" problem.

Dr. Harold V. Huneke
University of Oklahoma
Norman, Oklahoma

CRYPTARITHMS

TWO + THREE + SEVEN = TWELVE

A solution: 106 + 19722 + 82524 = 102352

Try this one:

CARTER + FORD = DEBATE

(Khajeh Nasseer Toosi-cont.)

3. A note on history: Aboo Jaafar Nasseer-ed-din Mohammad Ben Hassan Toosi (Khajeh Nasseer Toosi) was born in Toos, a city of Iran in about the year 1201 A.D. He wrote many papers and books, most of them in mathematics and astronomy. His paper on the parallel postulate gave Saccheri the basis for his geometry without parallels. Indeed, this work motivated Gauss and Lobachevski in the discovery of non-Euclidean geometries. We shall not go into the details of Toosi's contributions to mathematics. He discovered a mistake in Omar Khayyam's proof of the parallel postulate and tried to correct it. The result was his paper on the parallel postulate.

by
Ali R. Amir-Moez
John D. Hamilton
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Editor's Note: Below is a table for a five- line "poem" that uses these ideas.

K	I	N	H	P	G,Y	T	L,R,V	M,O	E,S	A
1	3	5	7	8	16	19	20	24	28	30

Vowels have been included, and, also, when a number has more than one corresponding letter, all of the letters are listed.

For example, to interpret your message you find that A corresponds to 30. The base two numeral for 30 is "11110_2" so A will appear in the second, third, fourth, and fifth lines but not in the first line.

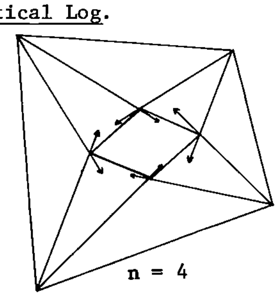
Since some of the letters appear more than once in a line of the poem, I have given you a super-big hint by indicating the number of letters and the spacing for each line.

The hard part in making these is picking a suitable "poem". Try this, it's fun! Then make one of your own.

line 1 _ _ _ _ _ _
line 2 _ _ _ _ _
line 3 _ _ _ _ _ _ _ _ _ _
line 4 _ _ _ _ _ _ _ _
line 5
_ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ !!

WEST'S CONJECTURE

If you trisect the angles of an n-gon, the figure formed by connecting the points of intersections of the trisecting rays that are adjacent to each side will be an equilateral n-gon. Editor's Note: Randy's conjecture is an extension of the theorem (for triangles) that was printed in the April, 1975, issue of The Mathematical Log.



n = 4

Randy West
James Madison Sr. High Sch.
Milwaukee, WI