THE MATHEMATICAL LOG

Volume XIX, No. 2 Winter 1974-5

CONSTRUCTING UNIFORM POLYHEDRA

A polyhedron is a set of plane figures enclosing a portion of three dimensional space. A polyhedron is uniform if all of its faces are regular polygons and all of its vertices are alike. In our world we are surrounded by polyhedra, and most people are familiar with such regular forms as cubes and pyramids, but few realize that many other beautiful figures actually have mathematical principles connected with them. My purpose in this paper is to describe the classes of uniform polyhedra and their members with figures and diagrams to show how each looks and also to map the faces for construction. All terms used in connection with polyhedra are derived from classical Greek as with polygons.

The five regular solids have congruent regular polygons for all faces. These regular solids are known today as Platonic solids since it has been known at least since Plato that there are only five possible. Euclid concludes his Elements with a proof that only five regular solids can possibly exist. The reasoning behind this proof goes as follows:

> ...each face shares one of its sides as a line in common with another face. These lines are called edges of the polyhedron. So each edge of a polyhedron belongs to exactly two faces and no more. The edges all meet at a point called a vertex to the polyhedron.
>
> In the tetrahedron three edges meet at each vertex, or to put it another way, each vertex is surrounded by three triangles. It is enlightening to lay out these three triangles flat and to notice the sum total of the number of degrees in the angles at a common vertex. Three sixties give 180 degrees. If a fourth triangle is introduced the total is 240 degrees, but now you have a vertex of the octahedron. Introducing a fifth triangle gives 300 degrees, and you have a vertex of the icosahedron. A sixth triangle gives 360 degrees, and you can see immediately that no polyhedral vertex arises. Everything stays flat.
>
> Next you try squares. A minimum of three is required, three 90's give 270 degrees, and a vertex of a cube can be formed. Adding a fourth square brings the total to 360 degrees and again you are left - flat. With pentagons the minimum of three will give you a vertex of the dodecahedron; four are too many, the total going beyond 360 degrees. With hexagons the minimum of three is already too many, three times 120 degrees. So no regular polyhedron exists with only hexagons as faces. And similarly for polygons with any greater number of sides.

The tetrahedron (fig. 1) has four equilateral triangular faces and is the simplest polyhedron since it has the least number of faces possible to enclose a three dimensional space. From a flat sheet of paper, a tetrahedron can be constructed from the following guidelines. The faces are numbered and dotted lines indicate folds here. In the drawing of the tetrahedron, the dotted line indicates the front view. (With the remainder of the solids, I will include these diagrams without explanation.) An interesting fact about the tetrahedron is that the centers of the four faces are the vertices of another tetrahedron and vice versa.[3]

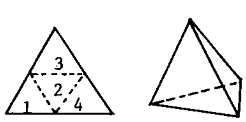


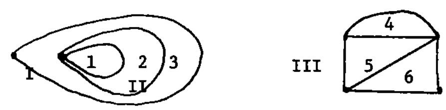
Figure 1

MATHEMATICS OF DOODLING

Have you ever dreamed that mathematics even gets into doodling? Well, it does, and you shall soon see how. Our doodles will be dealing with posts, which you may think of as points or dots; and fences, which you may think of as lines, not necessarily straight. Our doodles will have some restrictions. Every fence must do one of two things as shown:

1. Join 2 posts: Post  Post
2. Join one post to itself: Post 

If two fences cross, they may do so provided there is a post at the crossing. Every fence must have posts at its ends. We cannot have a fence without a supporting post. We shall also be talking about enclosures which fences generally make. For example, the following doodle has 6 enclosures.



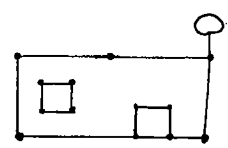
The above doodle has 3 connected pieces I, II, and III. Each connected piece has the property that one may move from any point of a fence or post of this piece to any other such point along fences and posts of the same piece.

Now let: P = The number of posts in the doodle
E = " " " enclosures in the doodle
F = " " " fences in the doodle
N = " " " pieces in the doodle

We are about to first conjecture a relation connecting P,E,F,N and then prove its correctness. For the above doodle of three pieces we get line A in the table:

	P	E	F	N
A	6	6	9	3
B	14	4	16	2

We get line B for the doodle



If we examine the table carefully we seem to have the following relation:

$$P + E = F + N.$$

At least, it holds for the two doodles we made. You may want to fill in the table for some doodles of your own and check to see if the relation still holds. In fact, we are about to show that the relation must hold for any doodle meeting our requirements.

Suppose we make a doodle having P posts. Remember, P may be a billion or even a googol. In making the doodle we could have first put in all the posts. Incidentally, each isolated post will be regarded as one connected piece. At this stage we have P posts, 0 enclosures, 0 fences and P pieces. The relation still holds: $P + 0 = 0 + P$

Let us assume that the formula holds at some stage in making the doodle, as it does when it has no fences. We now see whether the formula changes when we introduce our fences. Every time a fence is introduced, it falls into one of three possible cases.

case 1 The fence joins a post with itself
case 2 The fence joins two posts which are in the same piece
case 3 The fence joins two posts which are in different pieces

We are assuming that P + E = F + N. In cases 1 and 2, all that happens to the formula is we increase E by 1 and F by 1 which is the same as adding 1 to both sides. The formula still

continued on page 2, column 2

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WILL YOUR CHAPTER COMPETE IN THE MATH BOWL IN SEATTLE NEXT AUGUST?

Math Bowl is a five-round competitive math program patterned after the former TV show "College Bowl". Two teams of four students each attempt to answer questions covering a typical four-year college preparatory mathematics curriculum. The questions are flashed on a screen using an overhead projector. The first team to answer the question correctly is awarded ten points and is given a chance to answer a bonus question.

THIRD U.S.A. MATH OLYMPIAD

An article about the Third U.S.A. Mathematics Olympiad held in May, 1974, including the set of five problems used and their solutions is available in the January, 1975, Mathematics Teacher. This publication should be available through your teacher or at your school library. Look over these five interesting problems.

NATIONAL CONVENTION

The sixth national convention is scheduled for August 11-14, 1975, at Seattle University, Seattle, Washington. Information was mailed out in late January. If your chapter did not get the mailing, write to the general chairman:

Mr. Robert Marion
Route 1, Box 1080
North Bend, WA 98045

SECRETARY'S CORNER

Banner Price Increase

Our supply of Mu Alpha Theta banners was depleted, and the price for the new banners increased a great deal. They are now priced at $6.00.

Breakfast for Sponsors

A breakfast for sponsors is scheduled at the Denver meeting of NCTM for Friday morning, 7:30 a.m., April 25, 1975. It will be held in the Holiday Inn Downtown, and the cost is approximately $1.65 per person which includes tax and gratuity. You may send the money with the reservation, or we will collect it at the meeting. Since they have asked us for a guarantee, it is necessary to make reservations. Please send your reservation to the national office by April 15, 1975.

State Meetings

Several states are now well organized and are holding state conventions. In November, 1974, the Father Ryan High School chapter hosted the Tennessee meeting. In late January, the Louisiana State Convention is being held in Lafayette, and in February, the Warren T. White High School chapter is hosting the Texas State Convention.

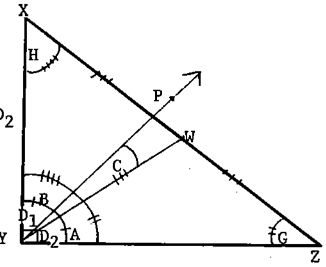
A PROBLEM SOLVED

Problem: Is there a relation between the angle formed by the median and the angle bisector of a triangle to the other angles of the triangle?

Find $\angle C$: (in terms of the other angles)

Given:

1. $\angle XYZ = 90^\circ$
2. $\overline{YW}$ bisects $\overline{XZ}$
3. $\overline{YP}$ bisects $\angle XYZ$
4. $\angle PYZ$ is $\angle D_2$
5. $\angle XYP$ is $\angle D_1$ } $\to \angle D_1 \cong \angle D_2$
6. $\angle XYW$ is $\angle B$
7. $\angle PYW$ is $\angle C$
8. $\angle WYZ$ is $\angle A$



$\overline{YW} = \overline{XW} = \overline{ZW}$ $\therefore$ ΔXWY is isoceles and ΔZWY is isoceles

$\angle H \cong \angle B$ and $\angle A \cong \angle G$

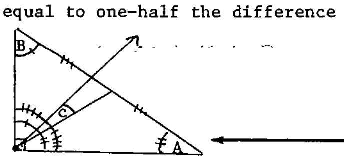
$\angle D_1 + \angle D_2 = 90^\circ$ and $\angle B + \angle A = 90^\circ$

$\angle D_1 + \angle C = \angle B \longrightarrow \angle B - \angle D_1 = \angle C$
$\angle A + \angle C = \angle D_2 \longrightarrow \angle D_2 - \angle A = \angle C$ } $\to$ Add the two equalities

$$\begin{array}{rl} & \angle B - \angle D_1 = \angle C \\ + & -\angle A + \angle D_2 = \angle C \\ \hline & \angle B - \angle A = 2\angle C \end{array} \longrightarrow B - A = 2C$$

$$\frac{B-A}{2} = \frac{2C}{2} \longrightarrow \frac{B-A}{2} = C$$

Theorem: In any right triangle, ΔABD, in which $\angle D$ is the right angle and $\angle A$ and $\angle B$ are opposite $\angle D$, the angle formed by the median of the hypotenuse and the angle bisector of $\angle D$ is equal to one-half the difference of $\angle A$ and $\angle B$.

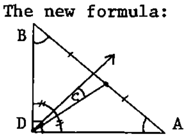


To apply this, the upper triangle is redrawn.

$\angle B$ is substituted for $\angle H$ because $\angle B = \angle H$
$\angle A$ is substituted for $\angle G$ because $\angle A = \angle G$

The angle formed by the median to the hypotenuse and the angle bisector of a right triangle is $\angle C$. The angles opposite the right angle are $\angle A$ and $\angle B$. According to the formula, $\frac{B-A}{2} = C$, "B - A" must be a positive number to obtain $\angle C$. Therefore, absolute value lines are placed around "B - A" in the event that "A" is larger than "B".

The new formula: $\frac{|B-A|}{2} = C$



$$\frac{|B-A|}{2} = C$$

by Arnold Jay Klein, Student at Mission Viejo High School, Mission Viejo, California

Editor's comment: Nice going, Arnold.

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holds after introducing fences of type shown by cases 1 and 2. For case 3, the only one left, we again increase F by 1 but this time reduce the number of pieces N by one. Adding 1 and then subtracting 1 on the same side does not change the value of that side. Our formula still holds. Since every doodle may be made by first putting in all the posts and then placing the fences, the formula must hold for every doodle satisfying the conditions we imposed.

If E = 0 and N = 1 the doodle may be called a tree. In this case P = F + 1.

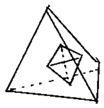
What is the formula if you try doodling on a doughnut? You may have to define "enclosure" more carefully.

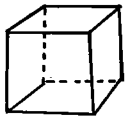
The formula we obtained is generally credited to Euler.

Harry D. Ruderman

continued from page 1, column 1

The cube (fig. 2) is a regular solid with six square faces. If sliced by a plane in a certain way, a cube can be cut so that its intersection with that plane forms a hexagon. (fig. 3)* The centers of the faces of a cube do not form the vertices of another cube, but an octahedron.





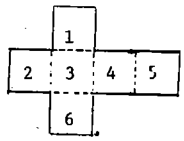


Figure 2

An octahedron (fig. 4) has eight equilateral triangular faces.

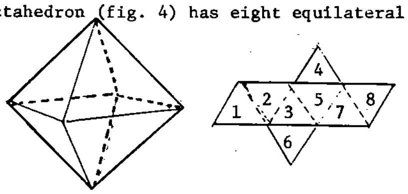
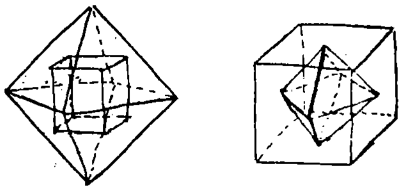
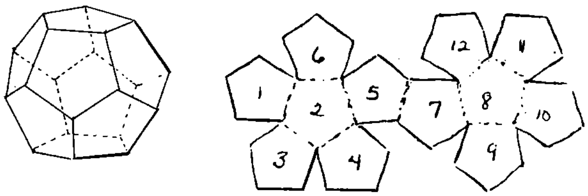


Figure 4

The centers of the faces of the octahedron forms the vertices of a cube, and conversely, the centers of the faces of a cube are the vertices of an octahedron.

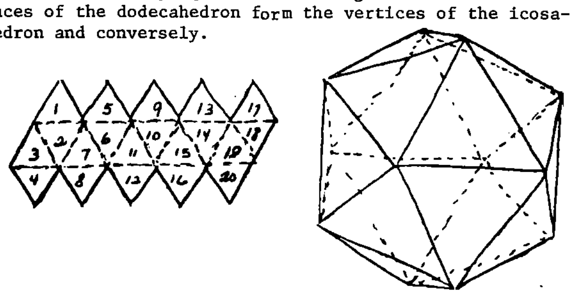


The next regular polyhedron is the regular dodecahedron. (fig. 5) It has twelve pentagonal faces.



A cube can be inscribed into a regular dodecahedron in such a manner that every edge of the cube becomes a diagonal of the dodecahedron. This can be done in five different ways.

The fifth regular solid is the icosahedron (fig. 6). It is composed of twenty equilateral triangles. The centers of the faces of the dodecahedron form the vertices of the icosahedron and conversely.



Another group of solids are known as Archimedian or semi-regular solids. These all have regular polygons as faces and all vertices are equal, but there are several types of polygons in each solid. There are thirteen such polyhedra and they are ascribed to Archimedes because he was the first to name and number them. Five of the thirteen semi-regular solids are formed from truncations of the regular solids. As defined before, portions of the regular solids are cut away.

The truncated tetrahedron (fig. 7) is formed by cutting the four triangles to form hexagons and adding then, four new triangular faces.

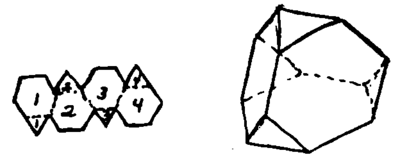
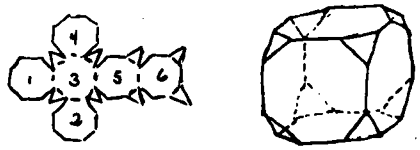


Figure 7

The truncated hexahedron or cube is formed by cutting the square faces to form octagons and adding eight new faces which are equilateral triangles.



The truncated octahedron (fig. 8) is formed from the octahedron by cutting the eight triangles to hexagons and adding six squares.

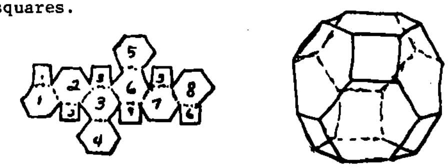
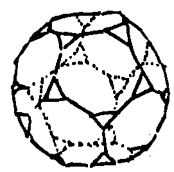


Figure 8

The truncated dodecahedron turns pentagons into twelve decagons and adds twenty triangles.



The truncated icosahedron (fig. 9) cuts triangles into twenty hexagons and adds twelve pentagons.

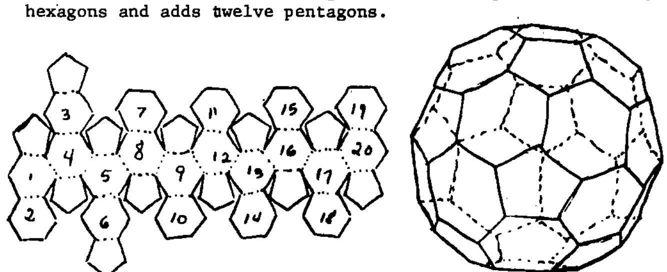


Figure 9

Another subset of the semi-regular solids is the quasi regular polyhedra. These solids are characterized by having two kinds of faces; each face of one kind is entirely surrounded by the other kind. They are the:
cuboctahedron (fig. 10) in which the six squares are on the facial planes of a cube and the eight triangles are on the facial planes of an octahedron;

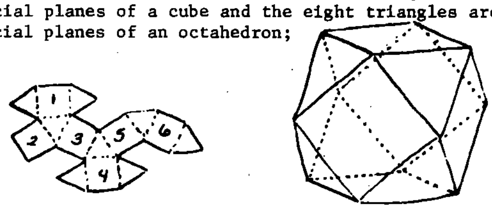
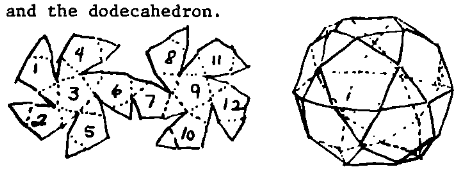


Figure 10

and the icosidodecahedron which is formed from the icosahedron and the dodecahedron.



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Two other semi-regular solids are the rhombicubectahedron (fig. 11) and the rhombicosidodecahedron.

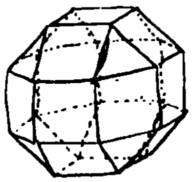
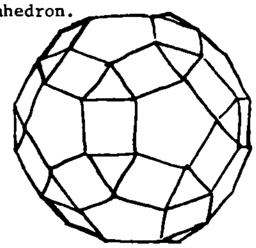
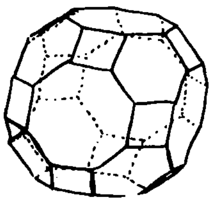
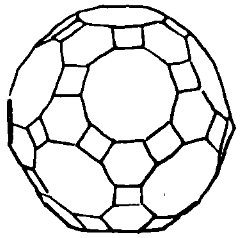


Figure 11

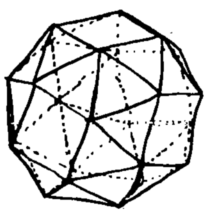


Two more are the truncated versions of the two preceding polyhedron: the rhombitruncated cuboctahedron and the rhombitruncated icosidodecahedron.

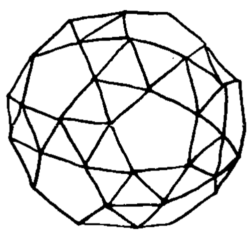




The final two are referred to as snub versions of the cube and the dodecahedron. The snub quality gives them a twisted appearance.

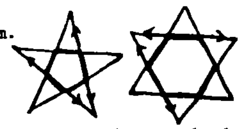


Snub cube (fig. 12)

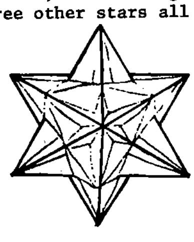


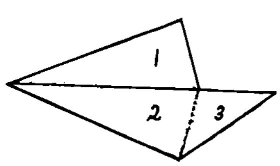
Snub dodecahedron

Extending non-adjacent faces of a figure until they intersect is called a stellation. When discussing stellations it is clearer to begin with two dimensional figures. Triangles have no non-adjacent sides and the extension of a square's non-adjacent sides are parallel. A pentagon is the simplest regular polygon on which a stellation can be performed. Extending the sides forms a common five-pointed star called a pentagram. The hexagon expands to a six-pointed star, and likewise, each polygon with five or more sides becomes a star when its sides are extended. Similarly, three dimensional polyhedra, with the planes of their faces extended, produce three dimensional stars. The tetrahedron and the cube, like the triangle and the square, are two exceptions.



From stellations of the octahedron and the dodecahedron, four regular stellated polyhedra can be formed. These four new polyhedra are the only regular stellated forms possible and, therefore, can be added to the list of uniform polyhedra. In the regular stellated polyhedra the facial planes are regular polygons and the faces meet at the edges by twos. The four regular stellated polyhedra are as follows:
The great icosahedron, which looks the most intricate of the four, has twelve points. Each vertex is constructed with five groups of three triangles. One of the three triangles, however, is not congruent to the remaining two. With the three other stars all triangles forming a point are congruent.





The small stellated dodecahedron (fig. 14) has twelve points with five triangular sides on each. The faces are in the form of a five-pointed star or pentagon.

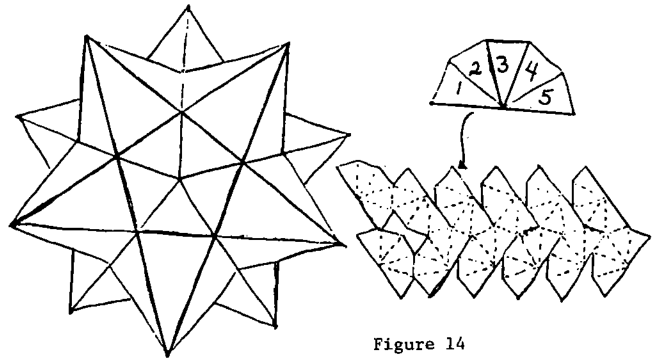
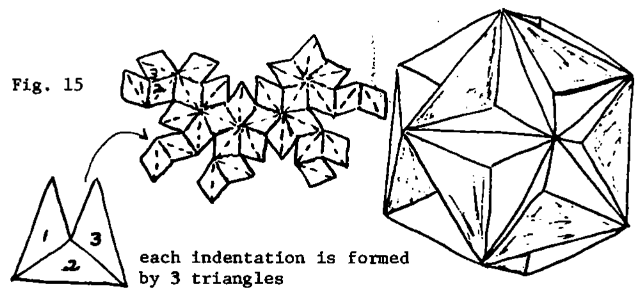


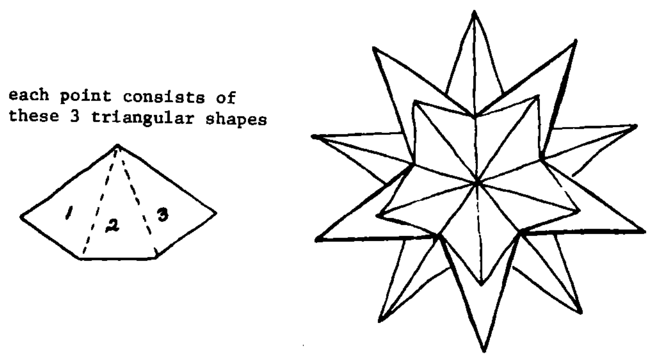
Figure 14

The great dodecahedron (fig. 15) has twelve non-convex points which form the faces of a pentagram star.

Fig. 15



The great stellated dodecahedron has twelve points, each with three triangular sides. The faces are pentagrams.



Another class of uniform polyhedra are the non-convex polyhedra. Badoureau in 1881 discovered thirty-seven more uniform polyhedra by a progress called faceting. Faceting is sort of the reverse of stellating in that portions of the solid are removed symmetrically.

In conclusion, there are seventy-five known uniform polyhedra from four different classes. Perhaps there are more; there is no proof that all have been discovered.

Karin Stahara, student
St. Louis University

[1]Magnus J. Wenninger, Polyhedron Models (Cambridge: Cambridge University Press, 1970) p. 1

[3]Hugo Steinhaus, Mathematical Snapshots (New York: Oxford University Press, 1969), P. 201

*Ibid., p. 170