THE MATHEMATICAL LOG

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AN EXCURSION INTO THE NON-EUCLIDEAN WORLD

Say there, you folks, did you ever meet anyone out of this world? Let me tell you! You could be out of this world, yourself, if you took off with me on a trip to the world of Henri Poincare. Further than the moon and closer than the stars, this non-Euclidean space has attractions that appeal to the majority of us who have been ravaged by a sub-zero winter, a spring freeze, a minus sunshine factor, and an energy crisis. Try to picture a land where there is no recession, no inflation, no depression. Fantastic, isn't it?

Before I tell you about this non-Euclidean World, I want to call your attention to the vagueness of such fundamental notions as straight line, plane and distance. Take, for example, the "distance between two points A and B." What does that mean? We think of measuring the distance, say with a foot rule, but that doesn't help much. A ruler is divided into equal intervals and hence presupposes the notion of distance. But what about equal distances? The distance, whatever it is, between A and B is equal to the distance between points C and D if the pair of points A and B can be shifted without changing their mutual distance, so as to coincide with the points C and D. But again the difficulty--the condition "without changing their mutual distance" implies that we know what is meant by equal distance.

Well, join the points A and B by a straight line. Surely we can say that the distance AB is equal to the distance CD, if the segment of the line AB can, by a rigid motion of this segment, be made to coincide with the segment joining C and D.

But does the notion of distance between two points necessarily involve the notion of straight line? Doesn't an intoxicated person cover distance between the bar and the swinging doors? And he certainly doesn't walk a straight line. And what is this rigid motion you speak about? Just think for a moment--isn't the notion of distance rather complicated? The imaginary world on which we are about to land will make us realize more vividly how serious these difficulties of distance really are.

This imaginary world is enclosed entirely within a large three-dimensional space whose boundary is a large spherical surface with radius R. The temperature here is rather fanciful. It changes from point to point. At the center it is a maximum and decreases gradually until it is absolute zero at the boundary. Suppose we are at a distance r from the center of the sphere, then t, the temperature is $(R^2 - r^2)$. When $r - 0$, at the very center, $t = R^2$, a maximum. When $r = R$, then $t = 0$, the temperature at the boundary. Another point: the temperature is constant on the surface of a sphere within our world provided this sphere is concentric with the world.

The inhabitants and material objects change with the temperature, growing larger or smaller in size in direct proportion to the change in temperature. Therefore, if a man walks toward the center he gets larger, towards the boundary he gets smaller. These changes, by-the-way, are instantaneous.

With this description in mind, we ought to begin to see some of the properties of the geometry which a man living in that world would develop. To him it would seem to be of infinite extent, although to us, it seems to be finite. As I mentioned before, if he started to walk toward the boundary, as the temperature fell, his body would grow smaller and his steps gradually shorter, contracting indefinitely as he approached the surface of the bounding sphere. To reach the boundary of his world, he would have to take an infinite number of steps, since $1/2 \log \frac{1 + x}{1 - x} \longrightarrow \infty$

The question naturally arises whether he would notice how bodies changed in size when their distance from the center of the world changed. The way in which we usually compare the sizes of objects is to place them side by side, or measure them, say with a ruler or tape. If he moved from one part of his world to another and took a beef burger with him, it would change at the same time and in the same proportion as he changed. It is obvious that the man would never realize that his beef burger is getting smaller. He is getting smaller himself.

In some respects his geometry would resemble our own, but in others it would differ. Suppose, for instance, the man wished to go from his house (H) to the dog house (DH) by the smallest number of steps. It is reasonable to suppose that the smallest number of steps would not be taken along a straight line HDH joining the house and the dog house, but rather along a path (say HnDH) which swerves toward the center, since his steps would be longer there. In fact, it can be rigorously proven, by the calculus of variations, that the path which gives the smallest number of steps is the arc of a circle which cuts the bounding sphere orthogonally. This fact has been proved previously in the literature. Such a circle we shall call a "shortest line" or "geodesic". The man would certainly think of it as our own "straight line." In another respect, this arc or shortest line has the same property as our straight line, namely, that between two points there can be only one straight line. Let us prove this.

Let us take a cross section of this sphere through center which will be the circle ABC. P is any point in the plane of the section and within the circle. Through P draw the diameter DD', also a line perpendicular to it through P, which will intersect the bounding circle in a pair of points, say T and T'. The tangents to the circle at T and T' will intersect upon the diameter, at a point which we call P'. This can easily be shown in plane geometry. (If two triangles have two sides of the one equal, respectively, to two sides of the other, and the angles opposite two equal sides equal, the angle opposite the other two equal sides are equal or supplementary, and if equal the triangles are equal.) Then any circle through P and P' will cut the given sphere orthogonally; and, conversely, any circle cutting the given circle orthogonally and passing through P will also pass through P'. It follows readily that through any two points within the sphere there is one and only one shortest line.

Proof:

> OTP' is by construction a right triangle, so that for any point P and the corresponding point P' we always have $OP \cdot OP' = OD^2$. Let QPP' be any circle through P and P', C its center, Q' the other extremity of the diameter through O. Then let QPP' cut circle ABC at M. Hence, $OM^2 = OQ \cdot OQ' = (OC - QC)(OC + QC) = OC^2 - QC^2 = OC^2 - MC^2$. Whence $OM^2 + MC^2 = OC^2$, and OMC is a right angle.

There is one respect, however, in which the geometry of the shortest line would differ from the geometry of our straight line.

From his parallel postulate, Euclid derived the theorem, that through a given point P not on a given line L, there is only one line parallel to L. One way of describing this notion is to draw any line through P cutting L in a point which we will call Q. If the point Q moves in either direction along L, the line PQ will at the same time revolve about the point P. The farther Q proceeds along L, in one of the two possible directions, the nearer will PQ approach a limiting position. Since there are two directions in which Q can move without limit along L, there are two ways of

continued on page 4

THE MATHEMATICAL LOG

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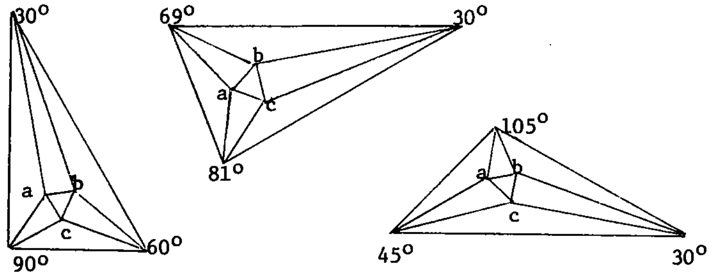
Calling Yahtzee fans...Ever wondered what your chances of getting a pair if you got three of a kind on the first roll? 1/4, 1/40 or 1/400?

You think you understand long numbers? Well then, how long would the answer to this evaluation be?

9^{9^9} We may assume you can write five digets to the inch. 30 in., 30 ft., 30 mi., none of these.

Here's one we're sure you didn't know! If you trisect the angles of a triangle (We know you can't do it by construction..use a protractor) and name the intersection of the rays that are adjacent to the side the triangle A,B,C, the triangle formed by connecting A, B, and C will always be equilateral. Honest. Try it. The property was first discovered by F. Morley around 1900, but it was not proven until 1914 by W.E. Phillips.



SIXTH NATIONAL CONVENTION-AUGUST, 1975

SEATTLE UNIVERSITY
SEATTLE, WASHINGTON

The program will feature Charles Allen, Director of NCTM, as Banquet Speaker. Other speakers include Roy Dubisch, Calvin Long, Dr. H.M. Cox, Elden Egbers and Jack Robertson. Some topics are: "How A Mathematician Thinks", "Serving PI", "How To Roll A Polygonal Wheel", "The M.A.A. Contest and the Olympiads", "Sperner's Lemma", "Coding and Matrices" and many, many more.

Special tours will include "Underground Seattle", The Pacific Science Center where we will hold a session, Pioneer Square and the Waterfront.

Plan NOW to come and bring a group. For additional details write: Robert Marion, Route 1, Box 1080, North Bend, Washington 98045.

SOME STUDENT TALKS FROM THE FIFTH NATIONAL MU ALPHA THETA MEETING

A FACTORIAL CURIOSITY

Dan Jasicki

Would you believe that there are more possible seating arrangements in a classroom with thirty chairs than there are drops of water in all the oceans of the earth?

The seating arrangment is found by

$$30! = 30 \cdot 29! = 30 \cdot 29 \cdot 28! \cong 2.65 \times 10^{32}$$

Let's agree the standard eyedropper is $1/10\text{cm}^3$.

$$1 \text{ drop} = 1/10\text{cm}^3$$

$$10 \text{ drops} = 1 \text{ cm}^3$$

$$10(100)^3 = 10^7 \text{ drops} = 1(\text{meter})^3$$

$$10^7(1000)^3 = 10^{16} \text{ drops} = 1(\text{kilometer})^3$$

Radius of earth is about 3959 miles or 6.37×10^3 kilometers.

If the earth were spherical its volume would be

$(4/3)\ r^3 = (1.333)(3.142)(6.37 \times 10^3)^3 = 1.08 \times 10^{12}$ cubic kilometers. Then the total number of drops would be...

$(1.08 \times 10^{12}) \times 10^{16} = 1.08 \times 10^{28}$ if the earth were all water!

Hey! Wait a minute! Our calculations show there are about 15,000 more seating arrangements than drops of water even if the whole world was water.

THE PERFECT SHUFFLE

Terry Ligocki

My topic deals with a "perfect" shuffle. By this I mean I take a deck of fifty-two cards and cut it into two piles of twenty-six cards each, then the bottom card of the bottom pack goes down first, then the bottom card of the top pack goes on top of that and so on (this will be demonstrated). One practical application of this is a line of cheerleaders [Dia. 1-(1)] who divide in half [Dia. 1-(2)], then both halves move toward one another [Dia. 1-(3)] and merge [Dia. 1-(4)]. In the third part [Dia. 1-(3)] the cheerleaders are turning over their signs and the letters on the right are the new letters which were originally face down.

If a deck has an even number (2n) of cards the new position of a card in the top half ($a \le n$) can be predicted by doubling its position (a) and subtracting one (2a-1). In the bottom half ($a > n$) a card's new position can be predicted by doubling its position (a) and subtracting the number of cards in the entire deck ($2a - 2n$ or $2(a - n)$ using the distributive property). As you will note, using the formula I have given you the first and last cards of any "even" deck don't move, so there are 2n - 2 "working" cards which is 2(n - 1) which is an even number.

If the number of cards in a deck is odd (2n - 1) then it can be divided in two ways--n cards on the top and n-1 on the bottom [Dia. 2-(A)] , or n-1 on the top and n on the bottom [Dia. 2-(B)]--which means it can be shuffled two ways (Dia.2).

To predict the position of a card in an "odd" deck, as in Dia. 2-(A) or-(B), the formula for the cards in the top "half" of the deck is the same as the one used for an "even" deck. The formula for the cards in the bottom "half" is 2(a-the number of cards in the top "half" of the deck (n or n-1)), except for the very bottom card in Dia. 2-(B) (labelled "e") which uses the special formula $[2 \cdot (a - n - 1)] - 1 = [2 \cdot (a - n + 1)] - 1 = (2a - 2n + 1 = 2(a - n) + 1$ because it is an "extra" card as you can see from the diagram.

In Dia. 2-(A) only the top card doesn't move so there are (2n - 1) - 1 "working" cards which is 2(n - 1) which is an even number. In Dia. 2-(B) the top card and the two very bottom cards don't move so there are (2n - 1) - 3 "working" cards which is 2(n - 2) which again is an even number of working cards.

continued on page 3

I leave it to you to decide why the number of "working" ards is always even, to explore other shuffles, and to preict the length and the number of loops cards go through--for ample with a deck of ten cards the loops are 2nd, 3rd, 5th, h, 8th, 6th,and back to 2nd,and 4th to 7th and back to 4th; en, with a deck of twelve cards there is only one loop ten ards long which is 2nd, 3rd, 5th, 9th, 6th, 11th, 10th, 8th, h, 7th, and back to 2nd.

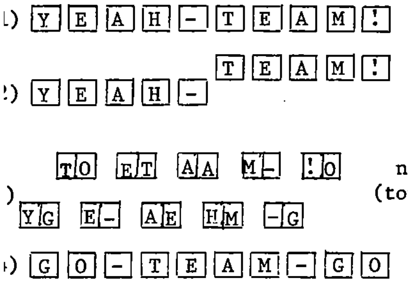


DIAGRAM 1

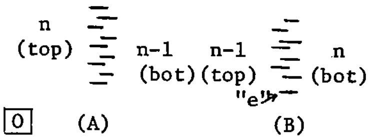


DIAGRAM 2

"1089"

Frank Gorenc

Take any three digit number whose digits don't repeat. Turn the number around and subtract the smaller from the larger. Take the number you get and turn it around and add them. You will get 1089.

Example:

$$\begin{array}{r} 321 \\ \underline{123} \\ 198 \\ \underline{891} \\ \text{"1089"} \end{array}$$

Proof:

$$\begin{array}{lr} & 100a + 10b + c \\ \text{(Subtract)} & \underline{100c + 10b + a} \\ & 99a - 99c = 99\cdot(a-c) \end{array}$$

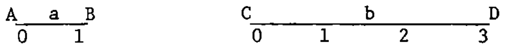
Let e = (a-c) then (100-1)e = 100e + 0 - e

$$\begin{array}{l} 100e + 0 - e + 100 - 100 \\ 100e - 100 + 90 + 10 - e \\ 100\cdot(e-1) + 10\cdot(9) + 1\cdot(10-e) \\ \underline{100\cdot(10-e) + 10\cdot(9) + 1\cdot(e-1)} \\ \qquad - 10 \\ 100e - 100 + 90 + 10 - e \\ \underline{1000-100e \quad + 90 \quad + e - 1} \\ \underline{1000 \qquad\quad + 80 \qquad\quad + 9} \\ \qquad \text{"1089"} \end{array}$$

Frank Gorenc

THE EQUIVALENCE OF THE UNIT LINE SEGMENT AND THE UNIT SQUARE

When dealing with infinite sets our usual intuitive feelings no longer hold. For example, we shall show that the line segment b which is 3 times as long as line segment a is "equivalent" to a. That is, we shall show that all

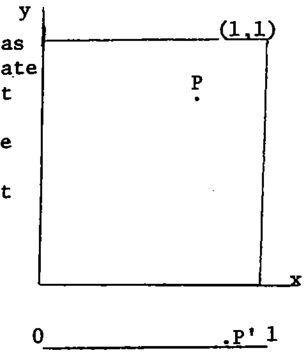


the points of a may be matched with all the points of b , in a one-to-one manner. Whenever there exists a one-to-one match or correspondence between two sets we say that the sets are equivalent. We are going to show that a and b are equivalent when regarded as sets of points. If P is any point of a it has associated with it a number n between 0 and 1: $0 \leq n \leq 1$. We say that the coordinate of P is n. The point P' having the coordinate 3n in b will be the point corresponding to P. It works the other way too. For any point in b we find its corresponding point in a by dividing its coordinate by 3 and locating the point having this number for its coordinate in a. The correspondence between the points of a and b may be shown by

n —— 3n

This correspondence proves that a and b are equivalent when regarded as sets of points. Of course, b could have been 10 units long, or 100, or a billion and a very similar proof can be made. In fact, a and b could have any lengths at all and in a similar manner we could show that a and b are equivalent when regarded as sets of points.

Now to show the equivalence of a unit line segment and a unit square. We have three parts to our presentation. In the first part the proof will have two flaws. You should stop and try to find them. In the second part we correct one of these flaws. In the third part we point out a fundamental flaw in our argument, one that you may have overlooked, and then show how it may be corrected. This will correct all the errors and the proof will be complete.



Part 1 We take our unit square located as shown in the first quadrant of a coordinate system. We show a unit line segment just below it. Let P be any point in or on the unit square. Denote its x-coordinate by $.a_1a_2a_3\ldots$ and its y-coordinate by $.b_1b_2b_3\ldots$ To find the point in the unit line segment, which we call P' corresponding to P , we find the coordinate of P' by intermeshing the x and y digits for P , getting

$$.a_1b_1a_2b_2a_3b_3\ldots$$

Now suppose that we want to find the point corresponding to any point P' in the unit line segment. Suppose that the coordinate of P' is

$$.c_1c_2c_3c_4c_5c_6\ldots$$

We simply reverse our meshing procedure to obtain the x and y coordinates:

$$x = .c_1c_3c_5\ldots \qquad y = .c_2c_4c_6\ldots$$

The mate of P' is P having (x,y) for its coordinates. Try to find the two flaws in the above argument.

Part 2 One of the difficulties we have to correct is that, except for 0, all terminating decimals have equivalent non-terminating decimals. For example:

$$.5 = .4999\ldots, \qquad .72 = .7199\ldots$$

When a coordinate has a terminating decimal we must decide whether to use it or its equivalent non-terminating decimal. To make things more uniform, we agree to always use the non-terminating form for this number, except for 0. For example,

In place of	We use
.5	.4999...
.72	.71999...
1.0	.999...
0.0	.000...

The only terminating decimal we are permitted to use is the one for 0, but even for 0 we shall use .000... With this agreement we have a unique way of obtaining the correspondence, supposedly. There is another flaw and a fundamental one, too. Have you found it?

Part 3 Strangely enough, with our unique way of naming numbers we actually have shown how to find the mate of every point in and on our unit square. It does not work from the line segment to the square. There are infinitely many points of the line segment that do not have mates in the square. Here is one example of a point that has no mate in the square:

.5101010101...

continued on page 4

If we split it we obtain for the x-coordinate .5000... which is unpermissible. We agreed to use in place of this numeral .499.... There are infinitely many such cases, but one is enough to show that we do NOT have, according to this approach, a one-to-one correspondence between the points of the unit square and the unit line segment. In fact, we do have a one-to-one correspondence between all the points of the unit square and a subset of the points of the unit line segment. To complete the proof we may invoke an important theorem which we shall state without proof. It is the Schroeder-Bernstein Theorem:

> If set A is equivalent to a subset of set B, and B is equivalent to a subset of A then sets A and B are equivalent.

Clearly, the unit line segment is equivalent to a subset of the square.

Our proof is complete. Do you see how the proof may be modified to show that a unit line segment is equivalent to a unit cube? Can you extend this further?

If you would like more information on infinite sets you may want to read one or more of the following:

"Theory of Sets" by Kamke, and translated by Frederick Bagemihl, published by Dover Publications, Inc. New York

"Introduction to the Theory of Sets" by Joseph Breuer, and translated by Howard F. Fehr, published by Prentice-Hall, New Jersey

"Naive Set Theory" by Paul R. Halmos, published by D. Van Nostrand

You need some help?

Yahtzee fans.....5/36 chances on the second roll, OK? and (31/36) (5/36) on the third roll for a probability of 335/1296 or approximately 1/4. Ohhh, you knew that!

$\log 9^{9^9} = 9^9 \log 9 = (387,420, 489)\cdot(.95424) = 369692127.4$
antilog $= 2.512 \times 10^{369692127}$

therefore 9^{9^9} has 369692127 digits in the answer and at the rate of 5 digits in an inch the answer would be 1167 miles long!! We knew you didn't know that.

The suggested proof for Morley's Theorem can be found on page 544 in Geometry-Fundamental Structure, Scott Foresman, 1965. (The complete solution is in the Teachers Commentary, page 115.)

AN EXCURSION INTO THE NON-EUCLIDEAN WORLD

obtaining limiting positions of the line PQ. Euclid's fifth postulate implies that these two limiting positions are the same, but, of course, our geometry need not be Euclidean.

Let us see what proposition the people in the "spherical world" would have corresponding to the proposition just quoted, if we think of the shortest lines playing for them the role of straight lines. We shall see that they would have no use for Euclid's postulate, and that as a statement of a general truth it would even appear ridiculous. Corresponding to the line L, they would have a circle L, meeting the boundary of the world at right angles. Through a point PP, not on L, they would construct another shortest line intersecting L at a point Q, then let Q travel off in either direction along L. For each position of Q there would be a definite circle PQ cutting the boundary orthogonally. Let us see what the limiting positions of PQ would be.

To the inhabitants of the sphere, Q would appear to be approaching the point R on the boundary, where L intersects it. But since their world would seem infinite to them, and no material object could ever reach its boundary, it would seem to them that the point Q could never reach R. They would consider PR a limiting position of PQ. By allowing Q to travel in the opposite direction, they would obtain a second limiting position PS. These two circles do not, in general, coincide. The angle θ, which they make with each other, is in general so appreciable that the inhabitants would easily observe it. Defining two shortest lines to be parallel, if they are in the same plane and do not meet, however far they are produced, we see that the inhabitants of this world would recognize the existence of an infinite number of shortest lines through P and parallel to L, namely, all such lines lying within the angle θ. The two limiting parallels PR and PS we call the principal parallels. On account of these contradictions to Euclidean geometry, we call this new geometry non-Euclidean.

Let us consider the angle θ. We notice that this angle decrease as the point P and the line L approach the center of the sphere, and may be made as small as we please by taking P and L sufficiently close to the center. We must remember that "sufficiently close" must be understood relatively to the radius of the bounding sphere. It is perhaps better to put it in this way: If P and L are at a given distance from the center, the angle θ decreases indefinitely as the radius of the bounding sphere is increased and may be made as small as we please by making this radius sufficiently large.

Now suppose our earth were at the center of such a sphere (concentric), and suppose the radius of the latter is assumed to be so large that in all the space about the earth which is accessible to our observation, the angle that above mentioned is smaller than any instrument of ours can detect.
The observations on the space in which we would thus find ourselves would differ in no particular from those to which we are accustomed; everything would look and feel just the same. But--and this is the point that is to be emphasized--the abstract non-Euclidean geometry may be applied to these observations quite as legitimately as the Euclidean. If we keep in mind the distinction between an abstract science and its concrete application, we see there is absolutely no way of telling whether we live in a Euclidean or in a non-Euclidean world. We do know that the angle θ, if it exists, is so small that we cannot detect it. So, for all practical purposes, Euclidean geometry is the more convenient to use, and there can be no doubt that it is the form of geometry which will always be used in practical applications. What we wish to emphasize is that the idea of there being more than one parallel to a line through a given point is in no way inconceivable, and in no way contradicts anything that we observe in our world. Henri Poincare, who thought of such a world, and Lobachewsky, who challenged the fifth postulate of Euclid, have given wonderful geometry to men. To study it is a rare experience.

You shall hear the question asked, "Which is the true geometry?" We can see now that the question cannot be answered for it is without meaning. It is very much as if we were to ask "Which is true, the measuring system using rulers marked off in feet and inches or marked off in centimeters?" One might ask whether it is more convenient to measure things in feet and inches than it is to measure them in centimeters. Non-Euclidean geometry is much more complex than Euclidean geometry, and as Poincare remarks, "Euclidean geometry is, and will remain, the more convenient." However, both are valid.

Whether our readers are in high school, college or in continuing education, the notions stressed in this paper are provocative. Young high school students should be introduced to the geometry of our fathers. However, there are many other ways of considering this branch of mathematics. For collegians a lecture sometime during the year will set at ease those students who have heard about non-Euclidean geometry. Keep in mind that all geometry is dependent on arbitrary (though reasonable) assumptions and hypothesis and that the changing of a postulate may produce different theorems.

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