THE MATHEMATICAL LOG

Vol. II, No. 1 September, 1958

ON THE RADIUS OF CURVATURE OF A LENS

A construction problem which leads to a useful physical application is the one of constructing the radius of a given sphere by the usual Euclidean method. This problem is left as an exercise for the reader but as a hint we give the application. The device for determining the radius of curvature of a spherical lens is called a spherometer. It consists of three parallel fixed legs whose perpendicular cross-section forms an equilateral triangle plus a fourth movable parallel leg which passes through the C. G. of the equilateral triangle. The movable leg is actually a calibrated micrometer screw which measures the distance from its bottom to the plane formed by the bottoms of the three fixed legs. To use the device one places it on a lens (the bottoms of the three fixed legs will be touching the surface). The screw is turned down until its bottom just touches the surface and a reading h is taken. The diameter of the lens follows from the following equations:

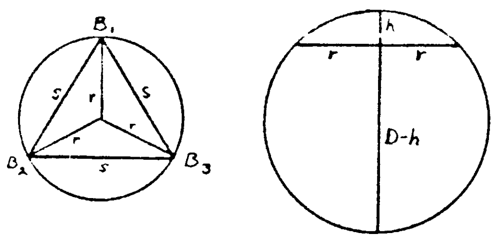
(B_1, B_2, B_3 are the bottoms of the three fixed legs).

$$r^2 = h(D - h)$$

$$r = \frac{s}{\sqrt{3}}$$

If $h << D$, then

$$D \cong \frac{s^2}{3h}$$



B_1, B_2, B_3 are bases of three fixed legs.

A PSYCHOLOGICAL GAME

The following game was invented by N. S. Mendelsohn and appears in the *American Mathematical Monthly*, vol. 53, 1946, pp. 86–88.

Club members might play the game and keep track of their results in order to compare with the theory outlined in the above article.

Two players A and B each choose a positive integer simultaneously; the player who calls the smaller number scores one point, unless the other player chooses a number just one unit greater, in which case the latter scores two points. Why should this be called a psychological game? This is because, in practice, each player attempts to guess his opponent's next move. After awhile, in spite of the fact that the game is equitable, one player will usually gain a psychological advantage over his opponent and pile up a high score.

A mathematical analysis of this game (not too easy) shows that the best strategy for player A is to call the numbers 1, 2, 3, 4, 5 with frequencies 1, 5, 4, 5, 1, the choice being made in random order. This strategy is best in the following sense: (1) if the game is played long enough no other strategy can beat it; (2) corresponding to any other strategy, a winning strategy can be devised; (3) any strategy which incorporates the number 6 or higher numbers would eventually be a losing strategy. —C. D. Olds.

TRAVELING LIBRARY

The traveling library is ready for you. Believing that a good selection of books is a wonderful luxury as well as a valuable tool, Mu Alpha Theta is proud to offer you the use of this library. A few chapters tried these books last spring and were enthusiastic.

Wouldn't your club activities be improved by the use of some modern mathematics references? The selection includes books on statistics, number theory, modern algebra, geometry, calculus, mathematical puzzles, Einstein theory, and integrated college mathematics. They offer lots of interesting material for meeting discussions or individual browsing. Many of these books are on the list you received last spring.

Our thanks to Professor Richard Andree for his valuable assistance and to the various publishers for their generous donations.

If you want to take advantage of this library, write to the librarian, Mr. George R. Hunt, Mathematics Dept., Odessa College, Odessa, Texas. Half a dozen books will be sent to you right away. On November 15 you will mail them to the next chapter on your list and report to Mr. Hunt by postcard. In turn, you will be sent a different set of books November 15, and about every six weeks.

Please send your request early so the best circuit for the books can be worked out.

MISCELLANEOUS TOPICS FOR CLUB TALKS

Finding Factors of Large Numbers. It would not take very long to find the prime factors of the number $N = 3354$. We would soon find by trial that this number is divisible by the primes 2, 3, 13, 43 so that $N = 2 \cdot 3 \cdot 13 \cdot 43$. A number like this one which is not itself a prime can always be factored uniquely into prime factors. If this is true, then to factor a given number we would only have to try dividing it in turn by the successive primes 2, 3, 5, 7, 11, 13, ...

This method of "trial and error" works nicely for numbers with four or five digits, but for larger numbers the work involved is prohibitive. These are two simple methods which can be used in finding the factorization of a number, provided it is not too large. The first method is due to Pierre de Fermat (1601–1665) and the second is due to Leonhard Euler (1707–1783). Both methods are explained in Chapter 4 of Oystien Ore's book *Number Theory and Its History*, New York, McGraw-Hill Book Company, 1948.

The Earth in a Balance. Have you ever wondered how man was able to weigh the earth? Of all the forlorn hopes of ever-hopeful man surely one of the seemingly impossible tasks was to attempt to put the earth in a balance and weigh it! However, man set his mind to this task and he succeeded. He has also been able to measure the diameter of the earth, the mass of the moon and its distance from the earth, the distance to the sun, and many other seemingly impossible measurements. An excellent club talk can be built around these topics. Useful material can be found in books of astronomy and elementary physics. In particular try to get a copy of the book by Herbert McKay: *The World of Numbers*, New York, The Macmillan Company, 1946.—C. D. Olds.

Our chapters now extend from Alaska to the Canal Zone, as well as from Maine to California.

THE MATHEMATICAL LOG

September, 1958 **Vol. II, No. 1**

The official publication of the National High School and Junior College Mathematics Club, Mu Alpha Theta. Address correspondence to Box 1127, The University of Oklahoma, Norman, Oklahoma.

A FAMOUS CONVERSE

A well-known theorem in geometry is this: *The lines which bisect the angles at the base of an isosceles triangle and which terminate in the opposite sides are equal.* The proof of this theorem is fairly easy, but the proof of the converse is quite difficult. Perhaps it is not even mentioned in your geometry text.

It would be interesting to see within the next year how many of your Club members can come up with proofs of the following converse.

Theorem. If the segments of the bisectors of the two base angles of a triangle terminated by the opposite sides are equal, then the triangle is isosceles.

The following proof due to Jacob Steiner (1796–1863), "the greatest geometrician since the time of Euclid," is well worth reproducing as a short Club topic. His proof was published in 1844 in *Crell's Journal*, vol. 28.

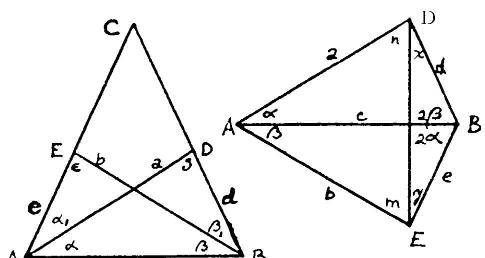


FIG. 1 FIG. 2

Given: $\alpha = \alpha_1$, $\beta = \beta_1$, AD = BE or a = b.

To prove: that $\triangle$ABC is isosceles, i.e. that $\alpha = \beta$.

Assume that $\alpha > \beta$,

Then in $\triangle$ADB and $\triangle$BEA we have side AB common, AD = BE, $\alpha > \beta$, hence BD > AE or d > e.

Hence $\alpha_1 + \alpha + \beta > \beta_1 + \beta + \alpha$, so $\delta > \epsilon$. (Why?)

Next place the triangles $\triangle$ADB and $\triangle$BEA as in Fig. 2. Here a = b. Draw the line DE. Then n = m. But $\delta > \epsilon$ so x > y and hence e > d.

But this is a contradiction since we have already shown that d > e. We conclude that $\alpha > \beta$ is false. A similar method shows $\beta > \alpha$ is false. Hence we must conclude that $\alpha = \beta$.

For other proofs see MATHEMATICAL GAZETTE, 1932, p. 200; 1933, p. 122; 1938, p. 365.

AMERICAN MATHEMATICAL MONTHLY, 1948, p. 495.—C. D. Olds.

CHAPTER LETTERS

It's always an inspiration to read letters from our energetic chapters. We wish space allowed quotations from all the fine letters received.

"I have been attending the Conference on Education, sponsored by the Tulsa public schools, and have gained so many new ideas for our club this fall. One is, just before Christmas, we will have a Mathematics exhibit, showing sidelights on mathematics, where the students will model, draw charts, make designs and write articles on 27 topics or more which will be most interesting and cause us all to learn many interesting sidelights to the mathematics we are studying in high school. Then we will exhibit some studies we will make on Advanced or Modern Mathematics, such as Finite Geometry, Non-Euclidean Geometry, Game Theory, Groups and Fields, Infinity, Logic, Number Theory, Space Travel and Ballistics, Statistics, Topology. . . ."

Mrs. Neva Gurley
Bristow, Oklahoma

"We have meetings twice a month, one person being responsible for the program. Each has tried to make his meeting the best. There has been a serious portion, some enrichment, found from extra reading and research, then supplied a recreation problem. We had explanation of the Binary system, which they ate up, further study of it, finally a trip to an excellent exhibit shown at the local branch of the I.B.M. corporation."

William D. Christian
Normandy, St. Louis, Mo.

Notre Dame chapter sent newspaper clippings and a program of their excellent science fair.

"Dr. Petersen, a professor of Mathematics at the University of New Mexico, gave an interesting talk on trick problems in mathematics. Last February we received an invitation to visit the New Mexico Institute of Mining and Technology. . . . We also made a brief tour of an ore crushing plant. The whole trip was very profitable."

Walta Neuner
Valley H.S., Albuquerque, N. M.

The banquet at Waverly, Tênn., must have been an outstanding event. The menu was dittoed and illustrated in three colors. If parabolas and ellipses turned out to be potato chips, what do you suppose they were eating when the menu listed "triangles and rectangles, circles, cube in a frustum, cardiod and pi"?

"We would like to order nine patches of the national seal. In our club we have nine seniors and would like to have patches for them to wear on their graduation robes."

Amos LeMaster
Ceredo-Kenova H.S., W. Virginia

The Mathematical Journal of Jesuit High School of Dallas, Tex., is an impressive publication with exceptionally well written articles on such topics as Probability, Analytic Geometry, Double Star Systems, Non-Euclidean Geometry, Mathematics in Insurance.

"We met for a 7:00 a.m. before school breakfast. All were armed with their slide rules. I had offered to pay the one's breakfast who solved first correctly a set of problems between the time the waitress took their order until their plates were set in front of them. One boy said, 'This is the first time I ever did*n't* want my order brought in a hurry.' "

Mrs. Blanche Moon
Bethany, Okla.

A FINITE GEOMETRY

Have you ever thought about the possibility of a geometry containing only a few points? Several such finite geometries are known. We shall examine one. We must agree to drop our conventional ideas of what are meant by "point" and "line" (undefined objects in geometry) and adopt the meanings given by a set of postulates. Although many sets of postulates might be formulated, the set which follows has been selected for its fruitfulness in interesting theorems. This set of postulates allows many of the theorems of ordinary plane geometry to be proved. You may find it interesting to construct a different set of postulates and definitions and see how many theorems may be proved using them.

Postulates

1. There are exactly twenty-five points in the geometry, and these points are designated by the twenty-five letters A, B, C, . . ., Y.

A	B	C	D	E
F	G	H	I	J
K	L	M	N	O
P	Q	R	S	T
U	V	W	X	Y

A	I	L	T	W
S	V	E	H	K
G	O	R	U	D
Y	C	F	N	Q
M	P	X	B	J

A	X	Q	O	H
R	K	I	B	Y
J	C	U	S	L
V	T	M	F	D
N	G	E	W	P

2. A line consists of five points located either in a row, or in a column of one of the three blocks above. Thus, there are exactly 30 lines in the geometry. Examples of lines are ABCDE, CHMRW, YCFNQ, and ASGYM. The points A, B, and J do *not* lie on the same line.

3. A line segment (point pair) is congruent to another line segment when *both* of the following conditions hold:
   (a) Both pairs occur in row lines, or both in column lines in our table above.
   (b) The number of (directed) steps between the points is the same in each pair. In counting directed steps between points, we shall always count to the right in rows, and down in columns, and the first letter of a row or column will be considered as following the last. Thus, in line ABCDE there are two steps from B to D, one from B to C and another from C to D. There are three steps from D to B (D to E, E to A, A to B). So AC and XJ and CS are all congruent, but AK is not congruent to AC. (Why not?) The number of steps between two points is the *distance* between the points. Either the directed or the undirected distances may be considered, but they should not be confused. It is sometimes desirable to count in the mod 5 system.

4. Two lines are parallel if they have no point in common.

5. A line l_1 is perpendicular to a line l_2 if and only if there exist two points z_1 and z_2 on l_1 such that for each point z on l_2, the absolute distance $|z_1z|$ is equal to the absolute distance $|z_2z|$. Thus the two lines ABCDE and AFKPU are perpendicular since we may take $z_1 = E$ and $z_2 = B$.

It is of interest to discover how many theorems of ordinary geometry may be obtained in this twenty-five-point geometry. It is easily shown by examination, and hence need not be postulated, that two distinct points determine a line, and that two distinct lines either intersect in one point, or are parallel. Three points not on the same line determine a "triangle." The points R, V, A all lie in the first column of block three, and are collinear. The points J, C, B determine a triangle. Furthermore, by postulate 3, the segments JC and BJ are congruent (each has "length" one step and both lie in rows). Hence the triangle JCB is an isosceles triangle. The triangle AST is scalene; the triangle HRL is equilateral.

Here are some of the theorems that hold both in ordinary Euclidean geometry and in this twenty-five-point geometry.

1. The three altitudes of a triangle meet at a point α.
2. The perpendicular bisectors of the three sides of a triangle meet at a point β.
3. The three medians of a triangle meet at a point μ.
4. The point of concurrence μ of the three medians of a triangle lies on a line joining the points α and β mentioned above, and divides it in the ratio of 2:1.

It is possible to develop an extensive theory of parallelograms and quadrilaterals in this geometry.

If, as in ordinary geometry, a circle is defined as the locus of all points at a given distance from a fixed point (center), many theorems may be obtained. A circle of radius AB and center A contains the six points B, E, I, W, X, H, and no others. The point F, for example, is *not* on this circle. The number of steps from A to F is 1, just as in AB, but the distances are not equal (i.e., the segments are not congruent) since AB lies in a row but AF lies in a column (see postulate 3).

The ellipse, hyperbola, and parabola all have counterparts in this geometry. Theorems such as "the locus of mid-points of a system of parallel chords of a parabola is a line perpendicular to the directrix," and "at the point where this locus meets the parabola, the tangent to the parabola is parallel to the system of chords" can be proved. It is even possible to work out a rigorous theory of area in this twenty-five-point geometry!

An interesting 121-point geometry is given in the paper "A Finite Geometry" by Alonzo Church, available from the Galois Institute of Mathematics at Long Island University, Brooklyn, New York. Finite geometrics have applications in group theory, number theory, and lattice theory.

From *Fundamentals of College Mathematics* by Brixey & Andree, Henry Holt Co., 383 Madison Ave., New York 17, N. Y.

MATHEMATICAL LOG PROBLEMS

17. If two points A and B are too close together, it is difficult to construct the perpendicular bisector in the usual way. The circles are usually of such small radius as to be hard to make accurately or the circles meet at such small angles that the intersections are not well-defined. How may such constructions be avoided?

18. Draw a circle, C, and mark its center, O. With compasses only find the vertices of a square inscribed in circle, C. (See January issue about Mascheroni Constructions.)

19. Find the least positive integer, made up of 1's only, that is divisible by the number $999\ldots9$ consisting of twenty-five 9's.

20. If n is an odd integer, then n^2-1 is always divisible by 8. Prove by mathematical induction.

21. Prove by mathematical induction that the number of straight lines that may be drawn connecting pairs of points in a set of n points all in the plane, no three of which lie in a straight line, is $\frac{1}{2}n\,(n-1)$.

22. If the sine of an angle is $(S^2-T^2)\,/\,(S^2+T^2)$, find the cosine of the angle. If S and T are integers, notice that these results enable us to determine right triangles whose sides are integral, like 6, 8, 10; 8, 15, 17; 5, 12, 13.

23. Prove that, in a trapezoid, the sum of the angles subtended by the smaller base is greater than the sum subtended by the larger base. Show in particular that, if the extended non-parallel sides meet in angle A and the parallel bases subtend this angle A at the intersection, then the smaller base subtends angle sum $180^\circ + A$ and the larger base subtends angle sum $180^\circ - A$.

24. A grocer has 10 boxes of 16 grapefruit. Each grapefruit is supposed to weigh 1 pound. The grocer has a scale which weighs to the nearest ounce. Can you in one weighing, without weighing all the grapefruit, tell the grocer which box of grapefruit has grapefruit each weighing only 15 ounces. Suppose there were two such short-weight boxes and you were given two weighings. Can this always be done?

25. A. If x and y are positive numbers, show $x^2 + y^2 \geqq 2xy$, with equality if, and only if, $x = y$.
    B. Show for all positive x, that $x + 1/x \geqq 2$. When does equality hold?
    C. Show for $0 \leqq A < 90^\circ$, that $\cos A + \sec A \geqq 2$. When does equality hold?

Vern Hoggatt
Problems and Solutions Editor
San Jose State College

The Problem Editor would like problems with solutions sent in from the various clubs to get a wide range of problem material.

MATHEMATICAL GAMES AND PUZZLES

Many mathematics clubs start or finish their meetings with a mathematical game or puzzle. A very interesting collection of such games can be found in the various issues of the *Scientific American* in the *Mathematical Games Department,* vol. 196 (1957), vol. 197 (1957). Your Public Library will have this magazine if it is not in your school library. Such games make for good fun on a club social evening.

Note the Bibliography at the back of each issue of the *Scientific American.* See also

1. J. A. H. Hunter, *Fun With Figures,* Oxford University Press, 1956.
2. Geoffrey Mott-Smith, *Mathematical Puzzles,* Dover Publications, Inc., 1954.
3. Maurice Kraitchik, *Mathematical Recreations,* Dover Publications, Inc., 1942.—C. D. Olds.

CHRISTMAS IDEA

The Sherman, Texas, mathematics club placed a tree in the front corridor and decorated it with geometric solids, painted and sprinkled with glitter. This is a tradition at Sherman High.

CHALLENGE RENEWED

"We are writing you to answer your challenge in the January, 1958, issue of "*Mathematical Log,*" "Can You Top This?" To our knowledge, Wicomico Senior High School owns all the books on the National Science Foundation list, all the mathematics books from the Traveling Science Library, and many others.

We are extremely proud, not only of our books, but of the circulation of them. To quote our librarian, Mrs. Branche Phillips: "Circulation of books was greater in the mathematics class than in any other class in the library the first semester."

Sincerely yours,
Anne Johnson,
Wicomico H.S.

Professor N. A. Court, one of the world's greatest living geometers, Mathematics Professor Emeritus at The University of Oklahoma, is the author of one of the books in our traveling library. This book has the rare distinction of having been translated into Chinese.

DICTIONARY

The Bristow, Oklahoma, chapter of Mu Alpha Theta has compiled and printed a splendid dictionary of the words and terms most likely to be encountered in the mathematics courses, grades 9 through 12, of Bristow High School. There are 46 pages of brief, but exact definitions. If you ever spent an hour hunting for a special definition, you appreciate what a service this chapter has performed.

Those beautiful 4-inch embroidered emblems are still available for only 65 cents.

TURKISH MATHEMATICAL EXAM

Below is a sample of an examination taken by Turkish high school students. It is a matriculation examination prepared by the Department of National Education of the Republic of Turkey, for eleventh grade students. Would you like to try it?

Scientific Section. 1956 September Questions.

No. 1 (30 Points) $y = \dfrac{(x-a)^2}{x^2 - 2x - 3}$

(A) Find a so that the slope of the tangent to the curve at $x = 2$ is 8/9.

(B) Draw its graph for $a = 1$.

(C) Find the range of values of m, for which $y = m$ cuts the graph of the function in (A) at 2 points. If these points are A and B, prove that the mid-point of $\overline{AB}$ will move on a straight line, when m changes.

(D) If $m = 1$, show that AOB angle is 90°.

No. 2 (15 Points)

$\dfrac{P(x)}{x-3}$ has a remainder of 5

$\dfrac{P(x)}{x-1}$ has a remainder of -1

$\dfrac{P(x)}{(x-3)(x-1)}$ has what remainder?

No. 3 (20 Points)
A parabola and a line D perpendicular to its axis are given. Any tangent drawn from a point M on the parabola and line D meets at P.
If point M on the parabola moves, the locus of K will be on the circle with center at F. Where F is the focus of the parabola and K is the foot of the perpendicular from P to the line through F and M.

No. 4 (20 Points) $x^2/225 + y^2/25 = 1$
M (9,4) is on the ellipse. The tangent drawn from M cuts the axis at TT′. The normal cuts the axis at NN′. If a parallel to the tangent is drawn through the origin O (0, 0) cuts the ellipse with two points. If one of the points is P, $\overline{MT}\ \overline{MT'} = \overline{MN}\ \overline{MN'} = \overline{OP}^2$. Show this.

No. 5 (15 Points)
If $A + B + C = \pi$
show that $\cos A + \cos B + \cos C = 1 + \sin A/2 \sin B/2 \sin C/2$.

Classical Section. 1956 September Questions.

No. 1 (35 Points) A (8, 0); B (0, 6); C (−4, 2)
A circle passes through A, B, and $\overline{AB}$ is the diameter of it. Write the equation of the circle which passes through C and is tangent to the first circle at O (0, 0).

No. 2 (25 Points)

Find the derivative of $y = \dfrac{x-3}{\sqrt{x^2 - 2x - 3}}$.

No. 3 (40 Points)

$y = \dfrac{x^2 - 4x + 4}{x^2 - 1}$ Draw its graph.

EVERYDAY USE OF GEOMETRICAL FORMS

You recognize common geometrical shapes everyday—square handkerchiefs, spherical balls, parallelogram parking spaces in a parking lot, ice cream cones, etc. Pictures of some of the unusual and striking geometrical forms would make a good bulletin board display. Contrast a common pyramid tent with the one pictured in *Life,* July 21, 1958. There are many interesting buildings. For instance, a huge hemi-spherical field house at Montana State College, a hyperbolic paraboloid Episcopal Church in Milwaukee, many catenary suspension bridges. If your members keep an observant eye on current magazines, your club will soon have a striking collection.—*J. Andree.*